§2 推迟势

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$$

$$\vec{B} = \nabla\times\vec{A}$$

势的基本方程

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B}=0$$

$$\nabla\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$$

$$\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}$$

$$\vec{B}=\nabla\times\vec{A}$$

$$\begin{cases}\nabla^2\varphi+\dfrac{\partial}{\partial t}\nabla\cdot\vec{A}=-\dfrac{\rho}{\varepsilon_0}\\[2ex] \nabla^2\vec{A}-\dfrac{1}{c^2}\dfrac{\partial^2\vec{A}}{\partial t^2}-\nabla\left(\nabla\cdot\vec{A}+\dfrac{1}{c^2}\dfrac{\partial\varphi}{\partial t}\right)=-\mu_0\vec{J}\end{cases}$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right) = -\mu_0 \vec{J}$$

$$\nabla^2 \varphi + \frac{\partial}{\partial t}\nabla\cdot\vec{A} = -\frac{\rho}{\varepsilon_0}$$

采用Lorenz规范：

$$\nabla\cdot\vec{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

采用Lorenz规范：

$$\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0$$

$$\left.\begin{aligned}&\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}\\&\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}\end{aligned}\right\}\quad\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\begin{bmatrix}A_1\\A_2\\A_3\\\varphi\end{bmatrix}=-\begin{bmatrix}\mu_0J_1\\\mu_0J_2\\\mu_0J_3\\\rho/\varepsilon_0\end{bmatrix}$$

5）达朗贝尔方程协变形式

在Lorenz 规范下，矢势和标势满足的方程：

$$\left.\begin{aligned}\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}&=-\mu_0\vec{J}\\ \nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}&=-\frac{\rho}{\varepsilon_0}\end{aligned}\right\}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}$$

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}$$

注意到：

$$\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\equiv\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu}=\square^2$$

$$x_4=\mathrm{i}ct$$

达朗贝尔方程方程可以改写成：

$$\square^2\vec{A}=-\mu_0\vec{J},$$

$$\square^2\varphi=-\mu_0c^2\rho$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J},$$
$$\Box^2 \varphi = -\mu_0 c^2 \rho$$

$$\Box^2 \vec{A} = -\mu_0 \vec{J},$$
$$\Box^2 \frac{\mathrm{i}\varphi}{c} = -\mu_0 \left(\mathrm{i}c\rho\right)$$

J_μ

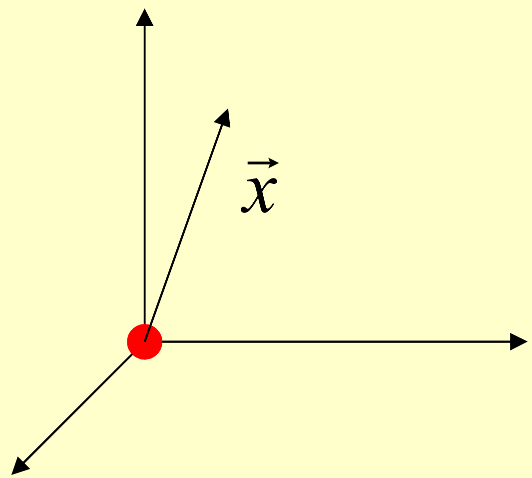
$$\left[A_\mu\right] = \left(A_1,\ A_2,\ A_3,\ \frac{\mathrm{i}}{c}\varphi\right)$$

达朗贝尔方程的协变形式：

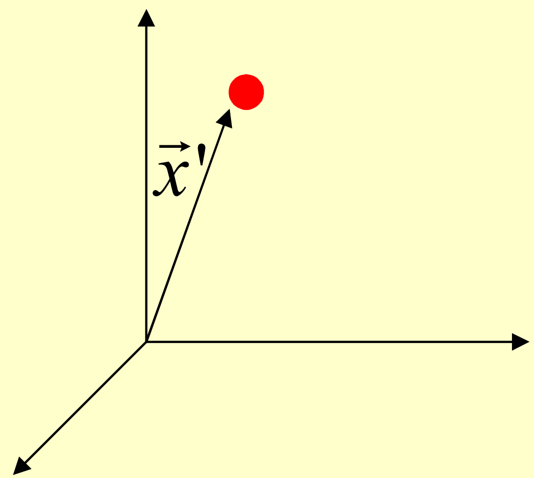
$$\Box^2 A_\mu = -\mu_0 J_\mu$$
$$\left(\mu = 1,2,3,4\right)$$

$$\Box^2 = \frac{\partial}{x_\mu}\frac{\partial}{x_\mu} = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

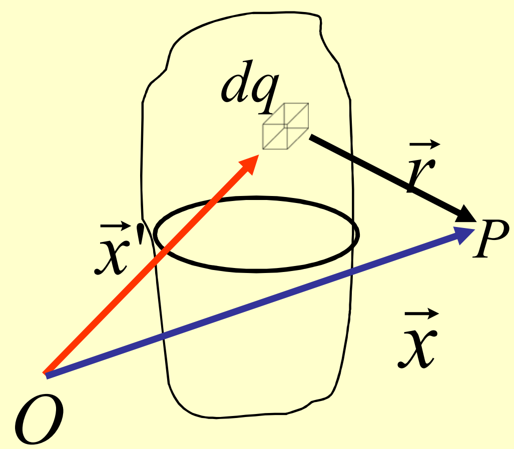
主要讨论的三点内容：



1. 任意一时刻、坐标原点处的含时点电荷的辐射势解



2. 任意一时刻、坐标 x' 处的含时点电荷的辐射势解



3. 一般的含时电荷分布的辐射势解的形式

1、辐射势满足的达郎贝尔方程

Lorenz（规范辅助）条件 $\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0$ 下

$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J}$$

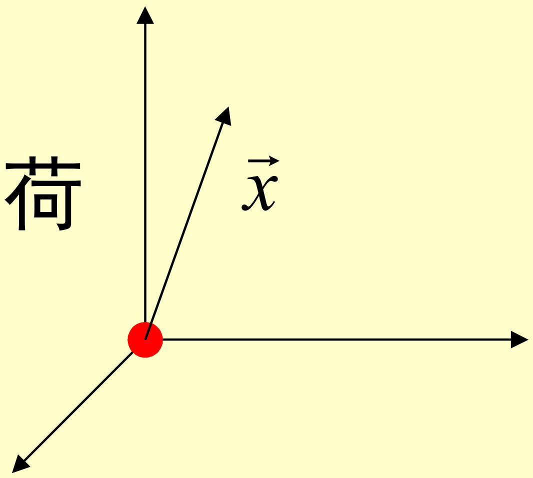
$$\nabla^2 \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

达郎贝尔方程：**线性方程、反映了电磁场的叠加性**

2、任意一时刻、坐标原点处的含时点电荷的辐射势解

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}$$

1）标势的波动方程：对于含时的点电荷



$$\rho(\vec{x},t)=Q(t)\delta(\vec{x})$$

根据体系的中心对称性：

$$\varphi(\vec{x},t)=\varphi(r,t)$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

$$\nabla^2 f=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

2）在原点之外，标势满足的波动方程为：

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=0 \qquad (r\neq 0)$$

——方程的解为球面波形式

$$\varphi(r,t)=\frac{u(r,t)}{r}$$

$$\frac{\partial^2 u}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0 \qquad (r\neq 0)$$

一维空间波动方程

$$\frac{\partial^2 u}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0 \qquad (r\neq 0)$$

一维空间波动方程的通解：

$$u(r,t)=f\left(t-\frac{r}{c}\right)+g\left(t+\frac{r}{c}\right)$$ （f、g为任意的两个函数）

令 $t'=t-\frac{r}{c}$

$$\Rightarrow \frac{\partial}{\partial r}f\left(t-\frac{r}{c}\right)=\frac{\partial}{\partial t'}f(t')\frac{\partial t'}{\partial r}=-\frac{1}{c}f'$$

$$\frac{\partial^2}{\partial r^2}f\left(t-\frac{r}{c}\right)=\frac{\partial}{\partial r}\left(-\frac{1}{c}f'\right)=\frac{1}{c^2}f''$$

$$\Rightarrow -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}f\left(t-\frac{r}{c}\right)=-\frac{1}{c^2}\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t'}f(t')\frac{\partial t'}{\partial t}\right]=-\frac{1}{c^2}f''$$

$$\frac{\partial^2 u}{\partial r^2}-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=0 \qquad (r\neq 0)$$

$$u(r,t)=f\left(t-\frac{r}{c}\right)+g\left(t+\frac{r}{c}\right)$$

3）除了原点之外，标势的通解形式：

$$\varphi(r,t)=\frac{f\left(t-\frac{r}{c}\right)}{r}+\frac{g\left(t+\frac{r}{c}\right)}{r}$$

向外发射的球面波

向球心收敛的球面波

$$\varphi(r,t)=\frac{f\left(t-\frac{r}{c}\right)}{r}+\frac{g\left(t+\frac{r}{c}\right)}{r}$$

$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

4）对于辐射问题，取： $g=0$

$$\varphi(r,t)=\frac{1}{r}f\left(t-\frac{r}{c}\right)$$

函数 f 的形式?

?

静电情况下： $\varphi(r)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$

推测一般情况下
[Q(t)为时间t的函数]

$$\varphi(r,t)=\frac{1}{4\pi\varepsilon_0}\frac{Q\left(t-\frac{r}{c}\right)}{r}$$

5）证明：$\varphi(r,t)=\dfrac{1}{4\pi\varepsilon_0 r}Q\left(t-\dfrac{r}{c}\right)$ 满足波动方程：

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi=-\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

---

r = 0为函数 $\left(\nabla^2-\dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\right)\left[\dfrac{1}{4\pi\varepsilon_0 r}Q\left(t-\dfrac{r}{c}\right)\right]$

的奇点。

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\left[\frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right)\right]=-\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

改写成

$$\nabla^2\left[\frac{1}{r}Q\left(t-\frac{r}{c}\right)\right]=\frac{1}{rc^2}\frac{\partial^2}{\partial t^2}Q\left(t-\frac{r}{c}\right)-4\pi Q(t)\delta(\vec{x})$$

$$=Q\left(t-\frac{r}{c}\right)\nabla^2\frac{1}{r}+2\left(\nabla\frac{1}{r}\right)\cdot\nabla Q\left(t-\frac{r}{c}\right)+\frac{1}{r}\nabla^2 Q\left(t-\frac{r}{c}\right)$$

$$\nabla^2\frac{1}{r}=-4\pi\delta(\vec{x}),$$

$$Q\left(t-\frac{r}{c}\right)\nabla^2\frac{1}{r}=-4\pi Q\left(t-\frac{r}{c}\right)\delta(\vec{x})=-4\pi Q(t)\delta(\vec{x}),$$

$$\nabla^2\left[\frac{1}{r}Q\left(t-\frac{r}{c}\right)\right]=-4\pi Q(t)\delta(\vec{x})+2\left(\nabla\frac{1}{r}\right)\cdot\nabla Q\left(t-\frac{r}{c}\right)+\frac{1}{r}\nabla^2 Q\left(t-\frac{r}{c}\right)$$

$$\nabla\frac{1}{r}=-\frac{1}{r^2}\nabla r=-\frac{1}{r^2}\vec{\mathrm{e}}_r$$

令 $t'=t-\frac{r}{c}$

$$\nabla Q\left(t-\frac{r}{c}\right)=\frac{\partial Q(t')}{\partial t'}\nabla t'=-\frac{1}{c}\frac{\partial Q}{\partial t'}\nabla r=-\frac{1}{c}\frac{\partial Q}{\partial t'}\vec{\mathrm{e}}_r$$

$$\nabla^2 Q=\nabla\cdot(\nabla Q)$$

$$=\nabla\cdot\left(-\frac{1}{c}\frac{\partial Q}{\partial t'}\vec{\mathrm{e}}_r\right)$$

$$\nabla\cdot\left(\varphi\vec{f}\right)=(\nabla\varphi)\cdot\vec{f}+\varphi\nabla\cdot\vec{f}$$

——I.19

$$=-\frac{1}{c}\left\{\nabla\left(\frac{\partial Q}{\partial t'}\right)\cdot\vec{\mathrm{e}}_r+\frac{\partial Q}{\partial t'}\nabla\cdot\vec{\mathrm{e}}_r\right\}$$

$$t' = t - \frac{r}{c}$$

$$\nabla^2 Q = -\frac{1}{c}\left\{\nabla\left(\frac{\partial Q}{\partial t'}\right)\cdot \vec{\mathrm{e}}_r + \frac{\partial Q}{\partial t'}\nabla\cdot\vec{\mathrm{e}}_r\right\} = \frac{1}{c^2}\frac{\partial^2 Q}{\partial t'^2} - \frac{2}{cr}\frac{\partial Q}{\partial t'}$$

$$\nabla\left(\frac{\partial Q}{\partial t'}\right)\cdot \vec{\mathrm{e}}_r = \frac{\partial^2 Q}{\partial t'^2}\nabla t'\cdot\vec{\mathrm{e}}_r$$

$$= -\frac{1}{c}\frac{\partial^2 Q}{\partial t'^2}\nabla r\cdot\vec{\mathrm{e}}_r = -\frac{1}{c}\frac{\partial^2 Q}{\partial t'^2}\vec{\mathrm{e}}_r\cdot\vec{\mathrm{e}}_r = -\frac{1}{c}\frac{\partial^2 Q}{\partial t'^2}$$

$$\frac{\partial Q}{\partial t'}\nabla\cdot\vec{\mathrm{e}}_r = \frac{2}{r}\frac{\partial Q}{\partial t'}$$

$$\nabla\cdot\vec{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta f_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial f_\phi}{\partial\phi}$$

$$\nabla^2\left[\frac{1}{r}Q\left(t-\frac{r}{c}\right)\right] = -4\pi Q(t)\delta(\vec{x}) + 2\left(\nabla\frac{1}{r}\right)\cdot\nabla Q\left(t-\frac{r}{c}\right) + \frac{1}{r}\nabla^2 Q\left(t-\frac{r}{c}\right)$$

$$\nabla\frac{1}{r} = -\frac{1}{r^2}\vec{\mathrm{e}}_r,$$

$$\nabla Q = -\frac{1}{c}\frac{\partial Q}{\partial t'}\vec{\mathrm{e}}_r,$$

$$\nabla^2 Q = \frac{1}{c^2}\frac{\partial^2 Q}{\partial t'^2} - \frac{2}{cr}\frac{\partial Q}{\partial t'}$$

$$= -4\pi Q(t)\delta(\vec{x})$$

$$+2\left(-\frac{1}{r^2}\right)\vec{\mathrm{e}}_r\cdot\left(-\frac{1}{c}\right)\frac{\partial Q(t')}{\partial t'}\vec{\mathrm{e}}_r \quad + \frac{1}{r}\left(\frac{1}{c^2}\frac{\partial^2 Q(t')}{\partial t'^2} - \frac{2}{cr}\frac{\partial Q(t')}{\partial t'}\right)$$

$$= -4\pi Q(t)\delta(\vec{x}) + \frac{2}{cr^2}\frac{\partial Q(t')}{\partial t'} + \frac{1}{r}\left(\frac{1}{c^2}\frac{\partial^2 Q(t')}{\partial t'^2} - \frac{2}{cr}\frac{\partial Q(t')}{\partial t'}\right)$$

$$= -4\pi Q(t)\delta(\vec{x}) + \frac{1}{rc^2}\frac{\partial^2 Q(t')}{\partial t'^2}$$

$$t' = t - \frac{r}{c}$$

$$\nabla^2\left[\frac{1}{r}Q\left(t-\frac{r}{c}\right)\right] = -4\pi Q(t)\delta(\vec{x}) + \frac{1}{rc^2}\frac{\partial^2 Q(t')}{\partial t'^2}$$

$$= -4\pi Q(t)\delta(\vec{x}) + \frac{1}{rc^2}\frac{\partial^2 Q(t')}{\partial t^2}$$

$$= -4\pi Q(t)\delta(\vec{x}) + \frac{1}{rc^2}\frac{\partial^2}{\partial t^2}Q\left(t-\frac{r}{c}\right)$$

或者

$$\nabla^2\left[\frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right)\right] - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right) = -\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\rho(\vec{x},t) = Q(t)\delta(\vec{x})$$

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{1}{\varepsilon_0}Q(t)\delta(\vec{x})$$

总结：任意一时刻、坐标原点处含时点电荷的辐射势解

$$\varphi(r,t) = \frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right)$$

任意一时刻、坐标原点处点电荷的辐射势解：

$$\varphi(r,t)=\frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right)$$

- 需要特别的注意：对势 $\varphi(\vec{x},t)$ 有贡献的不是同一时刻点电荷密度值，而是较早时刻 $t-r/c$ 的电荷密度值；

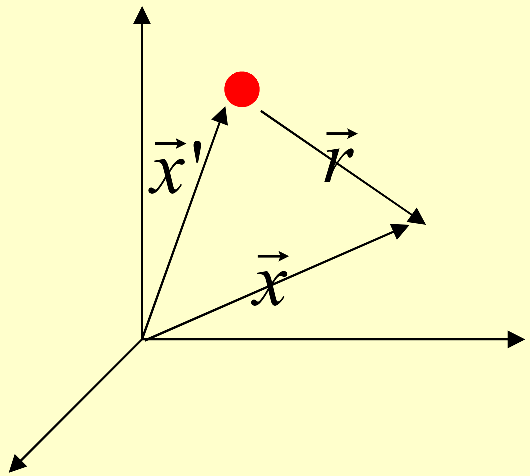
3、几个推论

任意一时刻、坐标原点处点电荷的辐射势解：

$$\varphi(r,t)=\frac{1}{4\pi\varepsilon_0 r}Q\left(t-\frac{r}{c}\right)$$

推论1：任意一时刻、位于 x' 处的点电荷的辐射标势

$$\varphi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0 r}Q\left(\vec{x}',t-\frac{r}{c}\right)$$



推论2：一般变化的电荷分布 $\rho(\vec{x}')$ 的辐射标势

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

推论2：一般含时的电荷分布 $\rho(\vec{x},t)$ 的辐射标势

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

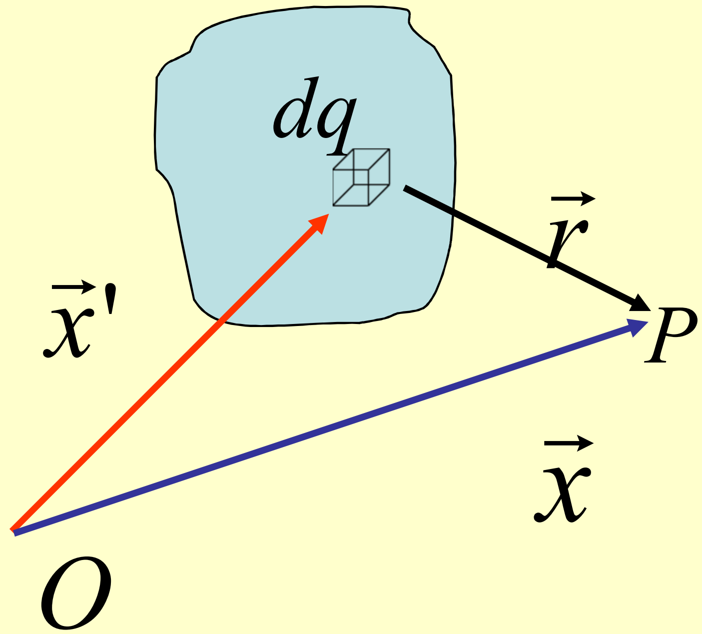
$$\nabla^2\varphi-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A}-\frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J}$$

推论3：一般变化电流分布 $\vec{J}(\vec{x},t)$ 的辐射矢势

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

推迟势的物理本质：



① 电荷产生的物理作用不能够立刻传至该场点，而是在较晚的时刻到达该场点；

② 这个推迟的时间为电磁作用传播所需要的时间；

③ 包括电磁作用在内的其它的一切作用，都是通过物质以有限的速度传播，不存在瞬时的超距相互作用。

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

④ Coulomb定律曾使人们认为电磁作用是瞬时作用。现在我们看到，这只是因为在静场条件下，t 时刻和$t-r/c$时刻的源没有差别，从而掩盖了推迟效应。

$$\rho(\vec{x}',t),\quad \vec{J}(\vec{x}',t) \Rightarrow \begin{cases} \varphi(\vec{x},t)=\int \frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V' \\ \vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int \frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V' \end{cases}$$

- 对于给定的电荷电流分布，先求出势，再通过下面的公式计算出电场和磁场

$$\vec{B}=\nabla\times\vec{A}$$

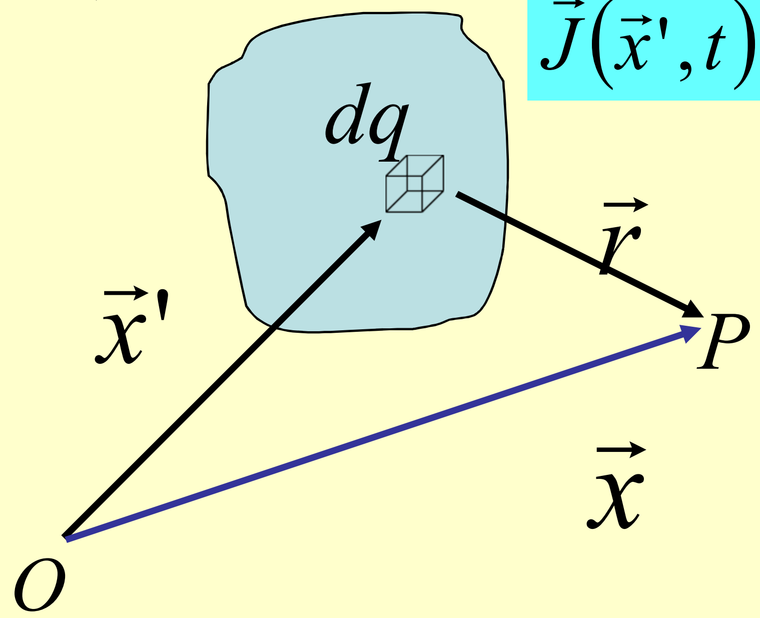
$$\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}$$

- 电磁场又反作用于空间的电荷电流分布。

- 给出了空间某点 $\vec{x}$ 在 时刻的势；

$\rho(\vec{x}',t),$

$\vec{J}(\vec{x}',t)$



$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

$$\varphi(\vec{x},t) = \int \frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

说明：上述公式的形式只对推迟势适用！

库仑定律（静电场）

$$\vec{E}(\vec{x}) = \int_{V'} \frac{\rho(\vec{x}')\vec{r}}{4\pi\varepsilon_0 r^3} \mathrm{d}V'$$

$$E(\vec{x},t) \neq \int \frac{\vec{r}}{4\pi\varepsilon_0 r^3} \rho(\vec{x}', t - \frac{r}{c}) \mathrm{d}V'$$

毕奥–萨伐耳定律（静磁场）

$$\vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}') \times \vec{r}}{r^3} \mathrm{d}V'$$

$$\vec{B}(\vec{x},t) \neq \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}', t - \frac{r}{c}) \times \vec{r}}{r^3} \mathrm{d}V'$$

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

验证：推迟势满足Lorenz（规范辅助）条件：

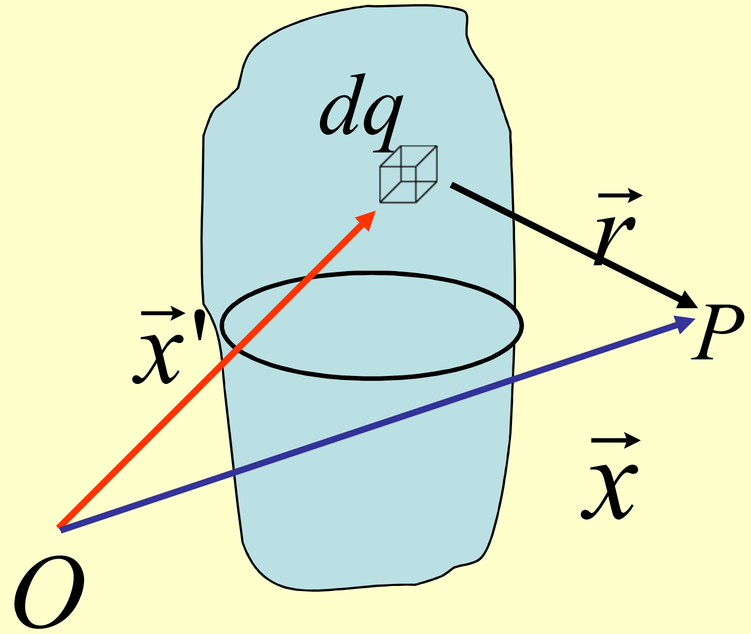
$$\nabla\cdot\vec{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0$$

$$\nabla = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

$$\nabla' = \vec{e}_x \frac{\partial}{\partial x'} + \vec{e}_y \frac{\partial}{\partial y'} + \vec{e}_z \frac{\partial}{\partial z'}$$

$$\boxed{r = \left|\vec{x} - \vec{x}'\right|} \qquad \nabla r = -\nabla' r$$

$$\nabla \frac{1}{r} = -\nabla' \frac{1}{r}$$



$$\boxed{t' = t - \frac{r}{c},} \qquad \nabla t' = -\nabla' t'$$

$$\frac{\partial t}{\partial t'} = 1$$

$$\nabla t' = -\frac{1}{c} \nabla r$$

$$\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\nabla\cdot\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\nabla\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]\mathrm{d}V'$$

$$\nabla\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]=\left(\nabla\frac{1}{r}\right)\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)+\frac{1}{r}\nabla\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)$$

$$\nabla\cdot\left(\varphi\vec{f}\right)=\left(\nabla\varphi\right)\cdot\vec{f}+\varphi\nabla\cdot\vec{f}$$

$$\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]=\left(\nabla'\frac{1}{r}\right)\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)+\frac{1}{r}\nabla'\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)$$

$$t' = t - \frac{r}{c}$$

$$\nabla \cdot \vec{J}(\vec{x}', t') = \frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \cdot \nabla t'$$

$$\nabla t' = -\nabla' t'$$

$$\nabla' \cdot \vec{J}\left(\vec{x}', t - \frac{r}{c}\right) = \nabla' \cdot \vec{J}(\vec{x}', t')\Big|_{t'\text{固定}} + \frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \cdot \nabla' t'$$

$$= \nabla' \cdot \vec{J}(\vec{x}', t')\Big|_{t'\text{固定}} - \frac{\partial \vec{J}(\vec{x}', t')}{\partial t'} \cdot \nabla t'$$

$$= \nabla' \cdot \vec{J}(\vec{x}', t')\Big|_{t'\text{固定}} - \nabla \cdot \vec{J}(\vec{x}', t')$$

$$\nabla' \cdot \left[\frac{1}{r}\vec{J}\left(\vec{x}', t - \frac{r}{c}\right)\right] = \left(\nabla' \frac{1}{r}\right) \cdot \vec{J}\left(\vec{x}', t - \frac{r}{c}\right) + \frac{1}{r}\nabla' \cdot \vec{J}\left(\vec{x}', t - \frac{r}{c}\right)$$

$$\nabla'\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)=\nabla'\cdot\vec{J}\left(\vec{x}',t'\right)\Big|_{t'\ \text{fixed}}-\nabla\cdot\vec{J}\left(\vec{x}',t'\right)$$

$$\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]$$

$$=\nabla'\frac{1}{r}\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)+\frac{1}{r}\nabla'\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)$$

$$=\nabla'\frac{1}{r}\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)-\frac{1}{r}\nabla\cdot\vec{J}\left(\vec{x}',t'\right)+\frac{1}{r}\nabla'\cdot\vec{J}\left(\vec{x}',t'\right)\Big|_{t'\text{固定}}$$

$$=-\nabla\frac{1}{r}\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)-\frac{1}{r}\nabla\cdot\vec{J}\left(\vec{x}',t'\right)+\frac{1}{r}\nabla'\cdot\vec{J}\left(\vec{x}',t'\right)\Big|_{t'\text{固定}}$$

$$\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]$$

$$=-\nabla\frac{1}{r}\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\;-\frac{1}{r}\nabla\cdot\vec{J}(\vec{x}',t')+\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}$$

$$\nabla\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]\;=\nabla\frac{1}{r}\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)+\frac{1}{r}\nabla\cdot\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)$$

$$=-\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]\;+\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}$$

$$\nabla\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]=-\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]+\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}$$

$$\Rightarrow\nabla\cdot\vec{A}(\vec{x},t)=\frac{\mu_0}{4\pi}\int\nabla\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]\mathrm{d}V'$$

$$=-\frac{\mu_0}{4\pi}\int\nabla'\cdot\left[\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\right]\mathrm{d}V'+\frac{\mu_0}{4\pi}\int\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}\mathrm{d}V'$$

$$=0$$

$$=\frac{\mu_0}{4\pi}\int\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}\mathrm{d}V'$$

$$\varphi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

另一方面： $\left(\nabla\cdot\vec{A}+\dfrac{1}{c^2}\dfrac{\partial\varphi}{\partial t}\right)$

$$t'=t-\frac{r}{c}$$

$$\frac{\partial\varphi(\vec{x},t)}{\partial t}=\int\frac{1}{4\pi\varepsilon_0 r}\frac{\partial}{\partial t}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$=\int\frac{1}{4\pi\varepsilon_0 r}\frac{\partial}{\partial t'}\rho(\vec{x}',t')\mathrm{d}V'$$

$$\nabla\cdot\vec{A}(\vec{x},t)+\frac{1}{c^2}\frac{\partial\varphi(\vec{x},t)}{\partial t}=\frac{\mu_0}{4\pi}\int\frac{1}{r}\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}\mathrm{d}V'$$

$$+\frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\frac{\partial}{\partial t'}\rho(\vec{x}',t')\mathrm{d}V'$$

$$\nabla\cdot\vec{J}(\vec{x},t)+\frac{\partial\rho(\vec{x},t)}{\partial t}=0$$

根据电荷守恒定律

$$\nabla'\cdot\vec{J}(\vec{x}',t')\Big|_{t'\text{固定}}+\frac{\partial}{\partial t'}\rho(\vec{x}',t')=0$$

因此

$$\nabla\cdot\vec{A}(\vec{x},t)+\frac{1}{c^2}\frac{\partial\varphi(\vec{x},t)}{\partial t}=0$$

即推迟势解满足洛伦兹规范。