Quantum Entanglement and Error Correction

Fall 2016

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Course Information

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Wechat Group: Course materials and discussions.

Download Scichat: http://scichat.com/

Scichat broadcast (beta): group number 2060

Please do NOT share the scichat video link (view in group please).

Registration and evaluation.

The Book

Part of the course will be based on the book

Quantum Information Meets Quantum Matter
– From Quantum Entanglement to Topological Phase of Matter

Bei Zeng, Xie Chen, Duan-Lu Zhou, Xiao-Gang Wen

In Springer Book Series -
Quantum Information Science and Technology

https://arxiv.org/abs/1508.02595

Quantum Mechanics in Finite Dimensional Systems

An arbitrary state can be expanded in the complete set of eigenvectors of $\hat{\mathbf{A}}$ ($\hat{\mathbf{A}}\Psi_i = a_i\Psi_i$).

$$\Psi = \sum_{i=1}^{n} c_i \Psi_i$$

When n is finite:

$$\Psi \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \Psi_1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Psi_2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \Psi_n \rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

A column vector. A basis.

Inner Products

$$\Psi \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \Phi \rightarrow \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix} \quad (\Psi, \Phi) = \sum_{i=1}^{n} c_i^* d_i$$

The matrix form:

$$(\Psi, \Phi) = \begin{pmatrix} c_1^* & c_2^* & \cdots & c_n^* \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

The conjugate transpose:

$$\Psi^\dagger = \Psi^{*T}, \quad (\Psi^\dagger)^\dagger = \Psi, \quad (\Psi, \Phi) = \Psi^\dagger \Phi$$

Example

$$\Psi = \begin{pmatrix} 1 \\ i \\ -1 \end{pmatrix} \quad \Phi \rightarrow \begin{pmatrix} 2i \\ 1 - i \\ 0 \end{pmatrix}$$

$$\Psi^\dagger = ?, \quad \Psi^\dagger \Phi = ?$$

Observables

$$\hat{\mathbf{A}}\Psi_i = a_i\Psi_i, \quad \hat{\mathbf{A}} \rightarrow \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

$$\hat{\mathbf{B}}\Psi_i = b_{ij}\Psi_j, \quad \hat{\mathbf{B}} \rightarrow \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

$$\hat{\mathbf{B}}^\dagger = \hat{\mathbf{B}}^{*T} \rightarrow \begin{pmatrix} b_{11}^* & b_{21}^* & \cdots & b_{n1}^* \\ b_{12}^* & b_{22}^* & \cdots & b_{n2}^* \\ \vdots & \vdots & \vdots & \vdots \\ b_{1n}^* & b_{2n}^* & \cdots & b_{nn}^* \end{pmatrix}$$

Observables

$$\Psi = \sum_{i=1}^{n} c_i \Psi_i$$

Average value:

$$(\Psi, \hat{\mathbf{B}}\Psi) \rightarrow \Psi^\dagger \hat{\mathbf{B}} \Psi = \langle \hat{\mathbf{B}} \rangle_\Psi$$

$$\begin{pmatrix} c_1^* & c_2^* & \cdots & c_n^* \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Dirac Notation

$$\Psi \rightarrow \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \rightarrow |\Psi\rangle \quad \text{ket}$$

$$\Psi = \sum_{i=1}^{n} c_i \Psi_i \rightarrow |\Psi\rangle = \sum_{i=1}^{n} c_i |\Psi_i\rangle$$

$$\Psi^\dagger \begin{pmatrix} c_1^* & c_2^* & \cdots & c_n^* \end{pmatrix} \rightarrow \langle\Psi| \quad \text{bra}$$

Inner Product

$$\Psi = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad \Phi = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{pmatrix}$$

$$(\Psi, \Phi) = \sum_{i=1}^{n} c_i^* d_i \to \langle \Psi | \Phi \rangle$$

Operators

$$\hat{\mathbf{A}}\Psi \rightarrow \mathbf{A}|\Psi\rangle = |\mathbf{A}\Psi\rangle$$

$$\langle \mathbf{A}\Psi| = \langle \Psi|\mathbf{A}^{\dagger}$$

The average value

$$\langle \hat{\mathbf{A}} \rangle_{\Psi} = \Psi^{\dagger}\hat{\mathbf{A}}\Psi \rightarrow \langle \Psi|\mathbf{A}|\Psi\rangle$$

A basis

$$\Psi_1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \Psi_2 \rightarrow \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad \Psi_n \rightarrow \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Psi \rightarrow |\Psi\rangle \rightarrow |i\rangle$$

Operators

A basis for matrix $|i\rangle\langle j|$

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \rightarrow \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} |i\rangle\langle j|$$

The identity operator

$$\mathbf{I} = \sum_{i=1}^{n} |i\rangle\langle i| = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example

For

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$$

and

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}(|0\rangle - i|1\rangle + |2\rangle)$$

and an operator

$$\mathbf{A} = i|0\rangle\langle 1| - i|1\rangle\langle 0| + |2\rangle\langle 2|$$

compute $\langle\psi_2|\psi_1\rangle$ and $\langle\psi_1|\mathbf{A}|\psi_1\rangle$.

Evolution

$$i\hbar \frac{\partial \Psi(\mathbf{r}, t)}{\partial t} = \hat{\mathbf{H}} \Psi(\mathbf{r}, t)$$

$$\rightarrow i\hbar \frac{\partial |\Psi(\mathbf{r}, t)\rangle}{\partial t} = \hat{\mathbf{H}} |\Psi(\mathbf{r}, t)\rangle.$$

If $\mathbf{H}$ is time independent

$$|\Psi(t_2)\rangle = \exp\left[\frac{-i\mathbf{H}(t_2 - t_1)}{\hbar}\right] |\Psi(t_1)\rangle = \mathbf{U}(t_1, t_2)|\Psi(t_1)\rangle$$

$$|\Psi\rangle \rightarrow \mathbf{U}|\Psi\rangle$$

Evolution is unitary

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$$

Two-level System

Electron Spin: The Stern-Gerlach Experiment

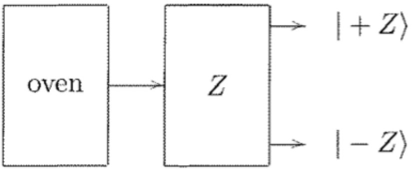

Figure 1.22. Abstract schematic of the Stern–Gerlach experiment. Hot hydrogen atoms are beamed from an oven through a magnetic field, causing a deflection either up ($|+Z\rangle$) or down ($|-Z\rangle$).

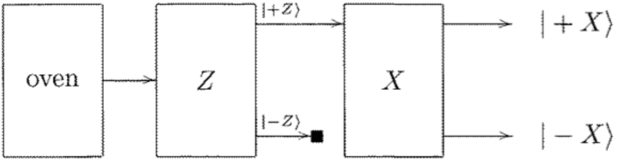

Figure 1.23. Cascaded Stern–Gerlach measurements.

Figure: From Nielsen & Chuang

The Stern-Gerlach Experiment

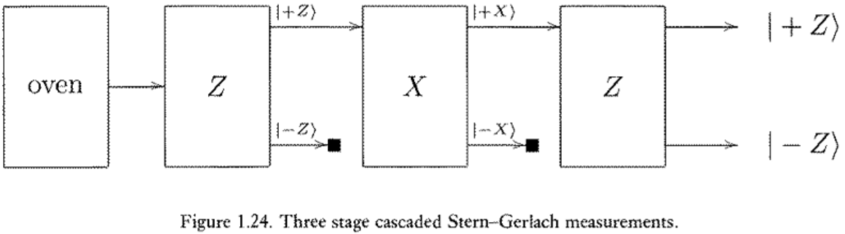

Figure: From Nielsen & Chuang

The interpretation: the electron spin, a two-level system

$$|+Z\rangle \leftarrow |0\rangle \qquad |+X\rangle \leftarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|-Z\rangle \leftarrow |1\rangle \qquad |-X\rangle \leftarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

Qubit

Two-level system $|0\rangle, |1\rangle$ – Qubit:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |\alpha|^2 + |\beta|^2 = 1.$$

$$|\psi\rangle = e^{i\gamma}\left(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle\right).$$

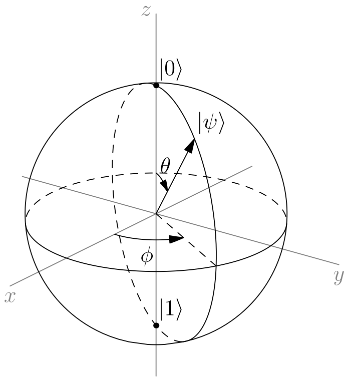

Figure: Bloch Sphere

The Pauli Matrices

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \sigma_x = \mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \sigma_y = \mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \sigma_z = \mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

form a basis for 2×2 matrices.

$$\vec{\sigma} = (\sigma_x \ \sigma_x \ \sigma_z)$$

Any 2×2 matrix $\mathbf{A}$ can be written as

$$\mathbf{A} = A_0\mathbf{I} + \vec{A} \cdot \vec{\sigma} = A_0\mathbf{I} + A_x\mathbf{X} + A_y\mathbf{Y} + A_z\mathbf{Z}$$

$$\begin{pmatrix} A_0 + A_z & A_x - iA_y \\ A_x + iA_y & A_0 - A_z \end{pmatrix}$$

Evolution of Qubit

A 2×2 unitary operator $\mathbf{U} = \begin{pmatrix} u_{00} & u_{01} \\ u_{10} & u_{11} \end{pmatrix}$ with $\mathbf{U}^\dagger \mathbf{U} = \mathbf{U}\mathbf{U}^\dagger = \mathbf{I}$ is a (single qubit) quantum gate

Example

- Bit flip: $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |0\rangle\langle 1| + |1\rangle\langle 0|$
- Phase flip: $\mathbf{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|$
- Hadamard Transform:
  $\mathbf{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|).$

The Density Operator

If with probability p_i we have the pure state $|\psi_i\rangle$, where $\sum_i p_i = 1$, we have an ensemble of pure states $\{p_i, |\psi_i\rangle\}$. We use the density operator ρ to describe this ensemble:

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

For a pure state $|\psi\rangle$, the corresponding density operator is $\rho = |\psi\rangle\langle\psi|$, and

$$\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = |\psi\rangle\langle\psi|\psi\rangle|\psi\rangle = |\psi\rangle\langle\psi|$$

The Density Operator

For the ensemble $\{p_i, |\psi_i\rangle\}$, we have the density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. For any observable $\mathbf{A}$, its average value is

$$\langle \mathbf{A} \rangle_\rho = \sum_i p_i \langle \psi_i | \mathbf{A} | \psi_i \rangle = \sum_i p_i \operatorname{tr}(\langle \psi_i | \mathbf{A} | \psi_i \rangle)$$

$$= \sum_i p_i \operatorname{tr}(\mathbf{A} |\psi_i\rangle\langle\psi_i|) = \sum_i \operatorname{tr}(\mathbf{A} p_i |\psi_i\rangle\langle\psi_i|) = \operatorname{tr}(\mathbf{A}\rho).$$

Properties of density operators

- Trace Condition

$$\operatorname{tr}(\rho) = \sum_i p_i \operatorname{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$$

- Positivity Condition: for any state ϕ,

$$\langle \phi | \rho | \phi \rangle = \sum_i p_i \langle \phi | \psi_i \rangle \langle \psi_i | \phi \rangle = \sum_i p_i |\langle \phi | \psi_i \rangle|^2 \geq 0.$$

The Density Operator

Example

For a single qubit, any density operator ρ can be written as

$$\rho = \frac{1}{2}(\mathbf{I} + \vec{r} \cdot \vec{\sigma}) = \frac{1}{2}(\mathbf{I} + r_x \mathbf{X} + r_y \mathbf{Y} + r_z \mathbf{Z})$$

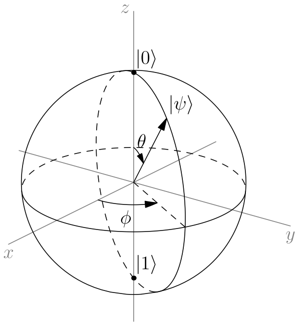

Figure: Bloch Sphere

Composite Systems

The state space of a composite physical system is the tensor product of the state spaces of the component physical systems

For two systems (two vector spaces) V_1 and V_2, we have two states

$$|\psi_1\rangle \in V_1, \quad |\psi_2\rangle \in V_2$$

The joint state of the total system is

$$|\psi_1\rangle \otimes |\psi_2\rangle$$

For n systems with states $|\psi_i\rangle$, $i = 1, 2, \ldots, n$

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

Composite Systems

Properties of tensor products

- bilinear

$$
\begin{aligned}
|\psi_1\rangle \otimes (\alpha|\psi_2\rangle + \beta|\phi_2\rangle) &= \alpha|\psi_1\rangle \otimes |\psi_2\rangle + \beta|\psi_1\rangle \otimes |\phi_2\rangle \\
(\alpha|\psi_1\rangle + \beta|\phi_1\rangle) \otimes |\psi_2\rangle &= \alpha|\psi_1\rangle \otimes |\psi_2\rangle + \beta|\phi_1\rangle \otimes |\psi_2\rangle
\end{aligned}
$$

- inner product

$$(\langle\psi_1| \otimes \langle\psi_2|)(|\phi_1\rangle \otimes |\phi_2\rangle) = \langle\psi_1|\phi_1\rangle\langle\psi_2|\phi_2\rangle$$

- operators $\mathbf{A}_1 \otimes \mathbf{A}_2$

$$(\mathbf{A}_1 \otimes \mathbf{A}_2)(|\psi_1\rangle \otimes |\psi_2\rangle) = (\mathbf{A}_1|\psi_1\rangle) \otimes (\mathbf{A}_2|\psi_2\rangle) = |\mathbf{A}_1\psi_1\rangle \otimes |\mathbf{A}_2\psi_2\rangle$$

Composite Systems

Matrix representation: Kronecker Product
For two matrices $\mathbf{A}$ $(m \times n)$ and $\mathbf{B}$ $(p \times q)$:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1q} \\ b_{21} & b_{22} & \cdots & b_{2q} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots & b_{pq} \end{pmatrix}$$

their tensor product

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} & \cdots & a_{1n}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} & \cdots & a_{2n}\mathbf{B} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}\mathbf{B} & a_{m2}\mathbf{B} & \cdots & a_{mn}\mathbf{B} \end{pmatrix} \quad mp \times nq$$

Kronecker Product

Example

For $\mathbf{X} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{Y} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$,

$$\mathbf{X} \otimes \mathbf{Y} = \begin{pmatrix} 0 \cdot \mathbf{Y} & 1 \cdot \mathbf{Y} \\ 1 \cdot \mathbf{Y} & 0 \cdot \mathbf{Y} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{X} \otimes \mathbf{Y} \quad \mathbf{X}_1 \otimes \mathbf{Y}_2 \quad \mathbf{X}_1\mathbf{Y}_2 \quad \mathbf{XY}$$

Multiple Qubits

A single qubit: $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$. $\{|0\rangle, |1\rangle\}$, a basis.
Two qubits: $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$, a basis.
Other notations:

$$|0\rangle \otimes |0\rangle \quad |0\rangle|0\rangle \quad |00\rangle$$

A two-qubit state:

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$$

For $|\psi_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and $|\psi_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$,

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

Two Qubit Gates

Example

$$\mathbf{X} \otimes \mathbf{Y}$$

$$\text{controlled-}\mathbf{NOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{controlled-}\mathbf{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Quantum Entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$|\psi\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle$$

$$|\psi_1\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \quad |\psi_2\rangle = \beta_0|0\rangle + \beta_1|1\rangle$$

Quantum Entanglement

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle)$$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \quad \text{eigenvectors of } \mathbf{X}$$

No-Cloning Theorem

W.K. Wootters and W.H. Zurek, A Single Quantum Cannot be Cloned, Nature 299 (1982), pp. 802?03.
Suppose we can do $|\psi\rangle \rightarrow |\psi\rangle|\psi\rangle$, then we have a unitary $\mathbf{U}$ acting on $|\psi\rangle|0\rangle$, such that

$$\begin{aligned} \mathbf{U}|0\rangle|0\rangle &\rightarrow |0\rangle|0\rangle \\ \mathbf{U}|1\rangle|0\rangle &\rightarrow |1\rangle|1\rangle \end{aligned}$$

Therefore,

$$\mathbf{U}(\alpha|0\rangle + \beta|1\rangle)|0\rangle \rightarrow \alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle$$

but we know that

$$\alpha|0\rangle|0\rangle + \beta|1\rangle|1\rangle \neq (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle)$$

The Reduced Density Operator

For a two-particle density operator ρ, the quantum state of the first particle is given by

$$\rho_1 = \operatorname{tr}_2 \rho$$

V_1 of dimension d_1, with basis $|a_1\rangle$, $a = 0, 1, \ldots, d_1 - 1$
V_2 of dimension d_2, with basis $|b_2\rangle$, $b = 0, 1, \ldots, d_2 - 1$

$$\begin{aligned} \operatorname{tr}_2(|a_1'\rangle\langle a_1|) \otimes (|b_2'\rangle\langle b_2|) &= |a_1'\rangle\langle a_1| \operatorname{tr}(|b_2'\rangle\langle b_2|) \\ &= |a_1'\rangle\langle a_1|\langle b_2|b_2'\rangle = |a_1'\rangle\langle a_1|\delta_{bb'} \end{aligned}$$

If $\rho = \sigma_1 \otimes \sigma_2$, then

$$\rho_1 = \operatorname{tr}_2 \rho = \operatorname{tr}_2(\sigma_1 \otimes \sigma_2) = \sigma_1 \otimes \operatorname{tr}(\sigma_2) = \sigma_1$$

The Reduced Density Operator

Example

For $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, find ρ_1.

$$\rho_1 = \operatorname{tr}_2 \rho = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \frac{\mathbf{I}}{2}$$

Why Partial Trace

Consider a two particle state $|\psi\rangle = a_{ij}|ij\rangle$, and an operator $\mathbf{A}_1$ which only acts on the first qubit. Then the average value

$$\begin{aligned}
\langle\psi|\mathbf{A}_1|\psi\rangle &= \left(\sum_{ij} a_{ij}^{*}\langle ij|\right)\mathbf{A}_1\left(\sum_{kl} a_{kl}|kl\rangle\right) \\
&= \sum_{ijkl} a_{ij}^{*}a_{kl}\langle i|\mathbf{A}_1|k\rangle\langle j|l\rangle = \sum_{ijk} a_{ij}^{*}a_{kj}\langle i|\mathbf{A}_1|k\rangle \\
&= \mathrm{tr}(\sum_{ijk} a_{ij}^{*}a_{kj}\langle i|\mathbf{A}_1|k\rangle) = \mathrm{tr}(\sum_{ijk} a_{ij}^{*}a_{kj}\mathbf{A}_1|k\rangle\langle i|) \\
&= \mathrm{tr}(\mathbf{A}_1\sum_{ijk} a_{ij}^{*}a_{kj}|k\rangle\langle i|) = \mathrm{tr}(\mathbf{A}_1\rho_1)
\end{aligned}$$