Open Quantum Systems

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Consider the system S is coupled with the environment E.
The evolution of the wave function $|\psi_{SE}\rangle \in \mathcal{H}_S \otimes \mathcal{H}_E$ is governed by the Shrödinger's equation

$$i\frac{\partial|\psi_{SE}(t)\rangle}{\partial t} = H_{SE}|\psi_{SE}(t)\rangle,$$

which is unitary

$$|\psi_{SE}(t)\rangle = U_{SE}(t, t_0)|\psi_{SE}(t_0)\rangle.$$

Or in terms of the density operator

$$\rho_{SE}(t) = U_{SE}(t, t_0)\rho_{SE}(t_0)U_{SE}^{\dagger}(t, t_0).$$

Question: what is the time evolution of the system state $\rho_S(t)$?

Kraus representation

Assume $\rho_{SE}(t_0) = \rho_S(t_0) \otimes |0_E\rangle\langle 0_E|$.

$$\begin{aligned}\rho_S(t) &= \mathrm{Tr}_E\, \rho_{SE}(t) \\ &= \mathrm{Tr}_E\, U_{SE}(t,t_0)(\rho_S(t_0) \otimes |0_E\rangle\langle 0_E|)U_{SE}^\dagger(t,t_0) \\ &= \sum_k \langle k_E|U_{SE}(t,t_0)|0_E\rangle \rho_S(t_0) \langle 0_E|U_{SE}^\dagger(t,t_0)|k_E\rangle.\end{aligned}$$

Write $E_k = \langle k_E|U_{SE}(t,t_0)|0_E\rangle$, then we get:

Kraus Representation for non-Unitary Evolution

$$\mathcal{E}(\rho_S(0)) = \rho_S(t) = \sum_k E_k \rho_S(t_0) E_k^\dagger,$$

where

$$\begin{aligned}\sum_k E_k^\dagger E_k &= \sum_k \langle 0_E|U_{SE}^\dagger(t,t_0)|k_E\rangle\langle k_E|U_{SE}(t,t_0)|0_E\rangle \\ &= \langle 0_E|U_{SE}^\dagger(t,t_0)U_{SE}(t,t_0)|0_E\rangle = I\end{aligned}$$

Kraus representation

$$\rho_S(t) = \sum_k E_k \rho_S(t_0) E_k^\dagger$$

- $\rho(t)$ is Hermitian:

$$\rho(t)^\dagger = \left(\sum_k E_k \rho(t_0) E_k^\dagger \right)^\dagger = \sum_k E_k \rho^\dagger(t_0) E_k^\dagger = \rho(t)$$

- $\rho(t)$ is with unit trace:

$$\mathrm{Tr}\, \rho(t) = \mathrm{Tr} \left(\sum_k E_k \rho(t_0) E_k^\dagger \right) = \mathrm{Tr} \left(\sum_k E_k^\dagger E_k \rho(t_0) \right) = 1$$

- $\rho(t)$ is positive:

$$\langle \psi | \rho(t) | \psi \rangle = \sum_k (\langle \psi | E_k) \rho(t_0) (E_k^\dagger | \psi \rangle) \geq 0.$$

Example

Orthogonal measurements $\{\Pi_k\}$:

$$\Pi_k = \Pi_k^\dagger, \quad \Pi_j \Pi_k = \delta_{jk} \Pi_k, \quad \sum_k \Pi_k = I,$$

then the quantum operation $\mathcal{M}$ describing the measurement is

$$\mathcal{M}(\rho) = \sum_k \Pi_k \rho \Pi_k.$$

When ρ is a pure state $|\psi\rangle$, the measurement will take $|\psi\rangle\langle\psi|$ to

$$\frac{\Pi_k |\psi\rangle\langle\psi| \Pi_k}{\langle\psi|\Pi_k|\psi\rangle},$$

with probability

$$p_k = \langle\psi|\Pi_k|\psi\rangle.$$

Master Equation

Shrödinger's equation

$$\frac{d\rho_{SE}}{dt} = -i[H_{SE}, \rho_{SE}].$$

Tracing out the environment:

$$\frac{d\rho_S}{dt} = \mathrm{Tr}_E(\frac{d\rho_{SE}}{dt}) = \mathrm{Tr}_E(-i[H_{SE}, \rho_{SE}]).$$

We only care about the system, we omit the subscript S.
We know that in general

$$\rho(t) = \mathcal{E}(\rho) = \sum_k E_k(t)\rho(t_0)E_k^\dagger(t).$$

We will want a differential equation for $\rho(t)$, which is not always possible.

Markov Approximation

$\rho(t+dt)$ is completely determined by $\rho(t)$:

$$\rho(t+dt) = \rho(t) + O(dt).$$

Expand the Kraus operators in terms of dt

$$\begin{aligned} E_0 &= I + (-iH + M)dt, \\ E_k &= \sqrt{dt} L_k, \;\; k > 0 \end{aligned}$$

where both H, M are chosen to be Hermitian and are zeroth order in dt, L_k are chosen to be Hermitian and are zeroth order in dt. The condition $\sum_k E_k^\dagger E_k = I$ the gives $M = -\frac{1}{2}\sum_{k>0} L_k^\dagger L_k$.

The Lindblad Equation

$$\frac{d\rho}{dt} = -i[H,\rho] + \sum_{k>0}(L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2}\rho L_k^\dagger L_k).$$

Master equations for a single qubit

Recall that $\vec{\sigma} = (X, Y, Z)$, and here we denote $\sigma_1 = X, \sigma_2 = Y, \sigma_3 = Z$. Then let

$$\sigma_\pm = X \pm iY.$$

To look at the interaction picture. Let

$$\tilde{\rho}(t) = e^{iHt}\rho(t)e^{-iHt},$$

which then gives

$$\frac{d\tilde{\rho}(t)}{dt} = \sum_{k>0}(\tilde{L}_k\rho\tilde{L}_k^\dagger - \frac{1}{2}\tilde{L}_k^\dagger\tilde{L}_k\rho - \frac{1}{2}\rho\tilde{L}_k^\dagger\tilde{L}_k),$$

where

$$\tilde{L}_k = e^{iHt}L_k e^{-iHt}.$$

Amplitude Damping

Spontaneous emission: two-level atom interacting with an electromagnetic environment.

$$H = H_S + H_E + V,$$

$$\begin{aligned} H_S &= \frac{\omega_a}{2}\sigma_z, \\ H_E &= \sum_j \omega_j b_j^\dagger b_j, \\ V &= \sum_j g_j(\sigma_+ b_j + \sigma_- b_j^\dagger). \end{aligned}$$

In the interaction picture:

$$\frac{d\tilde{\rho}}{dt} = -i[\tilde{V}, \tilde{\rho}],$$

where

$$\tilde{V} = \sum_j g_j(\sigma_+ b_j e^{-i(\omega_j - \omega_a)t} + \sigma_- b_j^\dagger e^{i(\omega_j - \omega_a)t}).$$

Amplitude Damping

The master equation of amplitude damping is given by

$$\frac{d\rho}{dt} = \frac{\Gamma}{2}(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-),$$

where Γ is the decay rate of the excited level.
Let $\gamma = 1 - e^{-\Gamma t}$, then one has

$$\rho(t) = \begin{pmatrix} \rho_{00} + \gamma\rho_{11} & \sqrt{1-\gamma}\rho_{01} \\ \sqrt{1-\gamma}\rho_{10} & (1-\gamma)\rho_{11} \end{pmatrix}$$

$$\rho(t) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger,$$

where the Kraus operators E_0, E_1 are given as the following.

Kraus Operators for Amplitude Damping

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}.$$

Amplitude Damping

The unitary picture U_{SE}

$$\begin{aligned} |0\rangle_S|0\rangle_E &\rightarrow |0\rangle_S|0\rangle_E \\ |1\rangle_S|0\rangle_E &\rightarrow \sqrt{1-\gamma}|1\rangle_S|0\rangle_E + \sqrt{\gamma}|0\rangle_S|1\rangle_E \end{aligned}$$

From the derivation of the Kraus representation we know that

$$E_k = \langle k_E|U_{SE}|0_E\rangle,$$

so we get

$$\begin{aligned} E_0 &= |0\rangle_S\langle 0|_S + \sqrt{1-\gamma}|1\rangle_S\langle 1|_S \\ E_1 &= \sqrt{\gamma}|0\rangle_S\langle 1|_S \end{aligned}$$

Then the probability for the atom keeping in the excited state is

$$\langle 1|\rho|1\rangle(t) = \frac{1-r_z(t)}{2} = e^{-\Gamma t}.$$

Phase Damping

The interaction

$$V = \sum_j g_j \sigma_z (b_j + b_j^\dagger).$$

The master equation can be simplified as

$$\frac{d\rho}{dt} = \Gamma[2\sigma_+\sigma_-\rho\sigma_+\sigma_- - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-],$$

where Γ is the decay rate from $|+\rangle$ to $|-\rangle$.
The Kraus operators

$$E_0 = \sqrt{1-\gamma} I, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

where $\gamma = 1 - e^{-\Gamma t}$.
The physical effect

$$\langle 0|\rho(t)|1\rangle \quad = \quad \langle 0|\rho(0)|1\rangle e^{-\Gamma t}.$$

Depolarizing

A two-level atom interacting with three independent reservoirs,

$$H = \sum_{j=1}^{3} H_{E_j} + V_j,$$

where $H_{E_j} = \sum_k \omega_{jk} b_{jk}^\dagger b_{jk}$, $V_j = \sum_k g_{jk}\sigma_j(b_{jk}^\dagger + b_{jk})$.
The master equation

$$\frac{d\rho}{dt} = \frac{\Gamma}{6}\sum_{j=1}^{3}(2\sigma_j\rho\sigma_j - \sigma_j\sigma_j\rho - \rho\sigma_j\sigma_j).$$

The Kraus operators

$$E_0 = \sqrt{1-\Gamma} I, \quad E_j = \sqrt{\tfrac{\Gamma}{3}}\sigma_j,\ j = 1, 2, 3.$$

The physical effect

$$\rho(t) = \rho(0)e^{-\Gamma t} + (1 - e^{-\Gamma t})\frac{I}{2}.$$