第七章

带电粒子和电磁场的相互作用

本章的思路

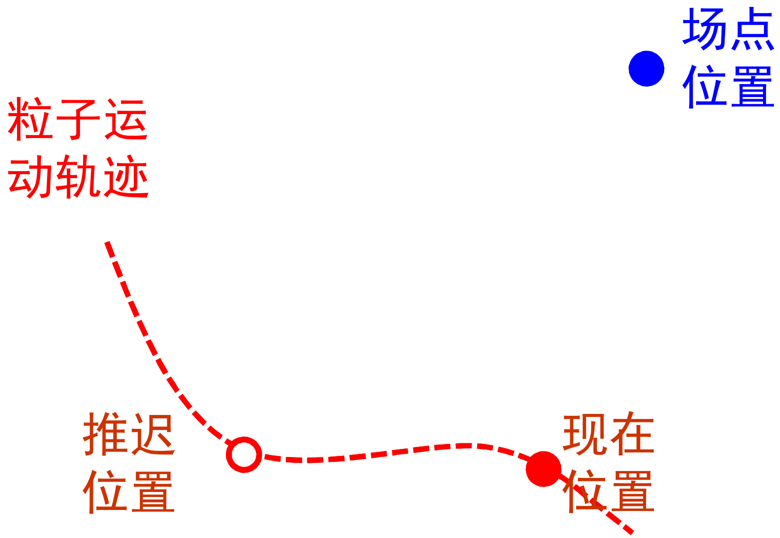
- 经典电动力学——**宏观介质**电磁现象规律；
- 本章思路：借助经典电动力学，**近似处理一个大的**带电粒子与电磁场的相互作用，没有考虑量子效应；
- 微观粒子：需要考虑量子效应

第一部分：单个带电粒子所激发的辐射电磁场

§1 运动带电粒子的势和辐射电磁场

§2 高速运动粒子的辐射

§4 切伦科夫(Cerenkov)辐射



$$\phi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

§1 运动带电粒子的势和辐射电磁场

1. 任意运动带电粒子的势——Lienard-Wiechert势
2. 偶极辐射
3. 任意运动粒子的电磁场

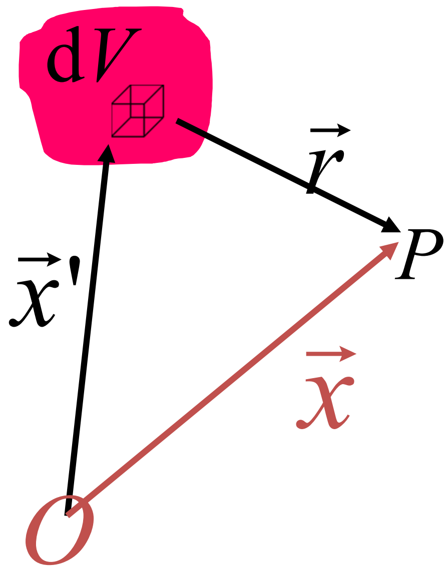
1、任意运动带电粒子的势

——Lienard-Wiechert势

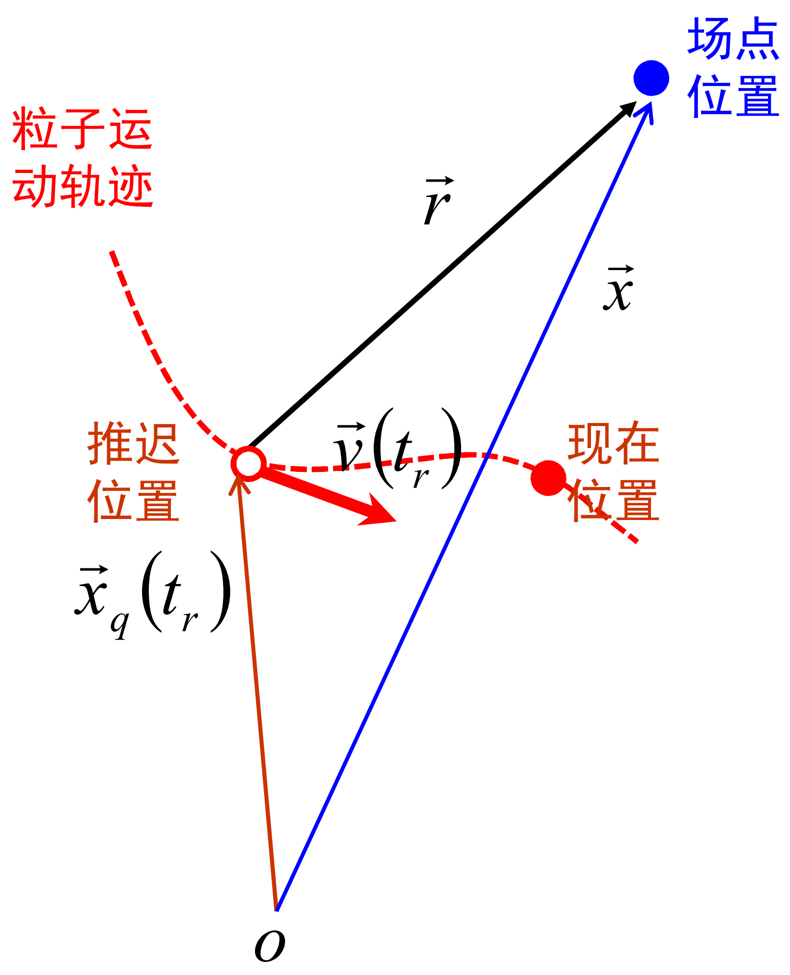
1）一般地，给定的电荷、电流分布 $\rho(\vec{x}',t)$, $\vec{J}(\vec{x}',t)$

$$\phi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$



- 对势有贡献的不是同一时刻源区各点的电荷、电流密度值，而是较早时刻 （t-r/c） 的电荷、电流密度值。



$$\phi(\vec{x},t)=\int\frac{1}{4\pi\varepsilon_0 r}\rho\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$\vec{A}(\vec{x},t)=\frac{1}{4\pi\varepsilon_0 c^2}\int\frac{1}{r}\vec{J}\left(\vec{x}',t-\frac{r}{c}\right)\mathrm{d}V'$$

$$t_r=t-\frac{r}{c},\ \left(t_r : t_{retarded}\right)$$

$$\vec{v}(t_r)=\frac{\mathrm{d}\vec{x}_q}{\mathrm{d}t_r}$$

- 假设：把运动的带电粒子看成是小体积内的电荷连续分布的极限：

$$\vec{J}\left(t-\frac{r}{c}\right)\mathrm{d}V'=qv\left(t-\frac{r}{c}\right)$$

v 为带电粒子在 t-r/c 时刻的速度。

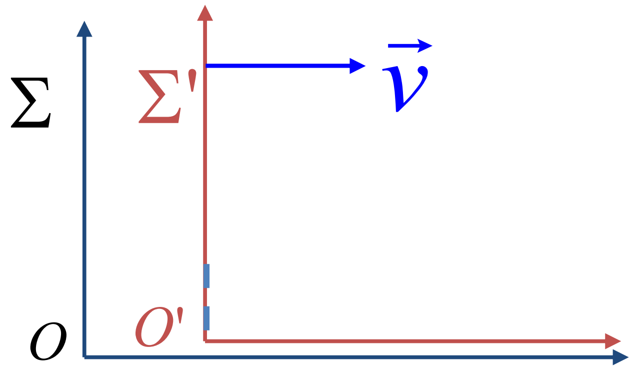
- 借助于惯性参照系之间四维势的变换，得到任意运动的带电粒子的势表达式。

$$A_4' = \mathrm{i}\frac{\varphi'}{c}$$

在静止参照系Σ'中，点电荷的势：

$$\vec{A}' = 0 \quad \varphi' = \frac{q}{4\pi\varepsilon_0 r'}$$

$$A_1' = A_2' = A_3' = 0, \quad A_4' = \mathrm{i}\frac{q}{4\pi\varepsilon_0 c r'}$$



根据四维矢量的Lorentz变换：

$$A = \tilde{a}\, A'$$

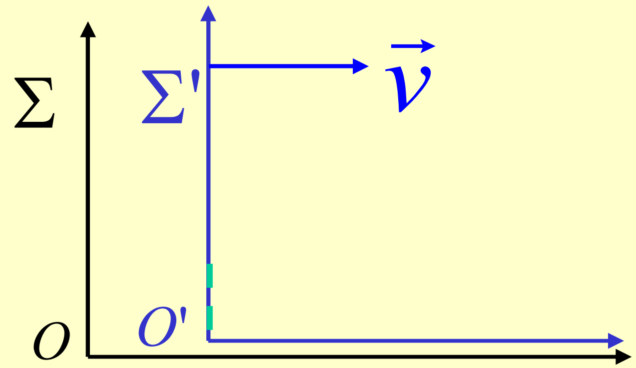
$$\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\mathrm{i}\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mathrm{i}\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \cdot \begin{bmatrix} A_1' \\ A_2' \\ A_3' \\ A_4' \end{bmatrix}$$

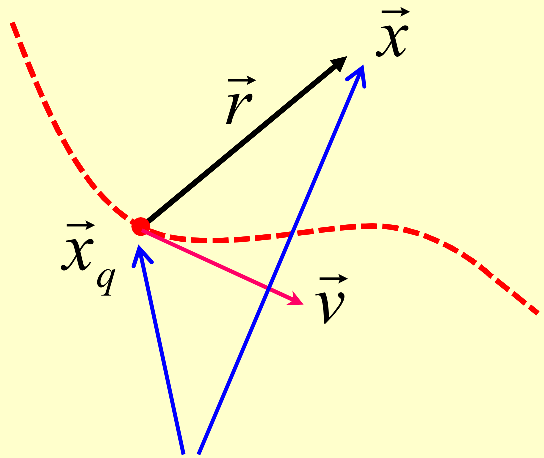
变换到 Σ 系，有：

$$\phi = \gamma \frac{q}{4\pi\varepsilon_0 r'}$$

$$A_1 = \gamma v \frac{q}{4\pi\varepsilon_0 c^2 r'}$$

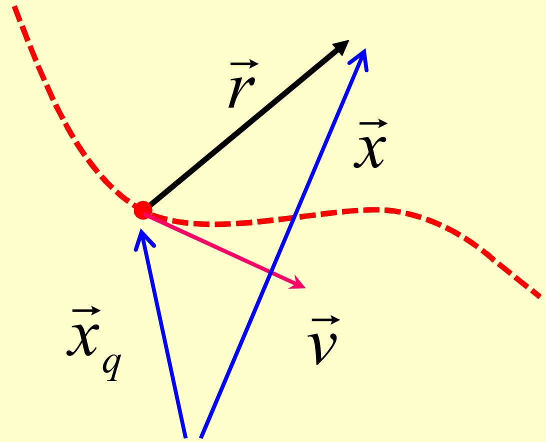
$$\vec{A} = \gamma \frac{q}{4\pi\varepsilon_0 c^2 r'} \vec{v}$$





余下步骤是将 *r*’ 用 Σ 系中的坐标 *r* 表示；

- **在Σ’ 参照系中：**
  电荷在某时刻 t1’ 产生的作用在 t2’ 时刻传播到场点所走过的距离



$$r' = c\left(t_2' - t_1'\right)$$

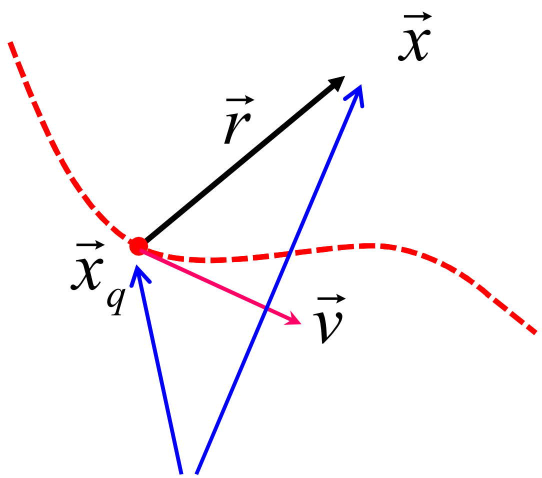
- **在 Σ 参照系中：**
  运动电荷的这种作用的产生和到达，分别发生在 t1 和 t2 时刻，则

$$r = c\left(t_2 - t_1\right) = \left|\vec{x} - \vec{x}_q\right|$$

- 当速度 v 和 x 轴成一定的角度时，空时坐标的Lorentz变换

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$t' = \gamma\left(t - \frac{\vec{v}\cdot\vec{x}}{c^2}\right)$$



- 在两个参照系中电荷的作用从发出-到达的距离存在如下的关系

$$r' = \gamma\left[c(t_2 - t_1) - \frac{1}{c}\vec{v}\cdot(\vec{x} - \vec{x}_q)\right]$$

$$r' = c(t_2' - t_1')$$

$$r = c(t_2 - t_1)$$

$$r' = \gamma\left(r - \frac{1}{c}\vec{v}\cdot\vec{r}\right)$$

$$\phi = \gamma \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}$$

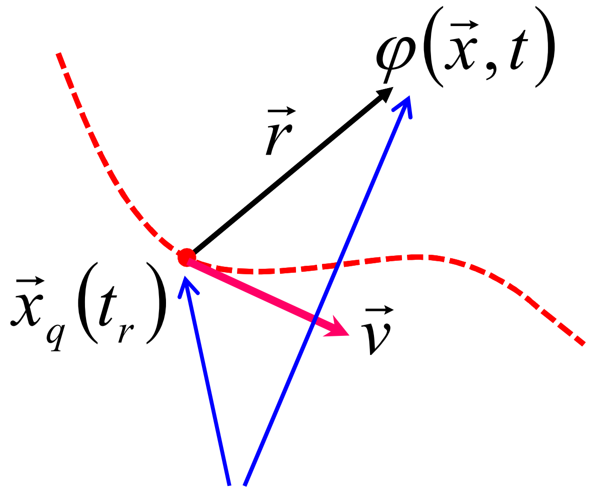
$$\vec{A} = \gamma \frac{1}{4\pi\varepsilon_0 c^2}\frac{q\vec{v}}{r'}$$

$$r' = \gamma\left(r - \frac{1}{c}\vec{v}\cdot\vec{r}\right)$$

李纳-维谢尔(Liénard-Wiechert)势:

$$\phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r - \frac{1}{c}\vec{v}\cdot\vec{r}}$$

$$\vec{A}(\vec{x},t) = \frac{1}{4\pi\varepsilon_0 c^2}\cdot\frac{q\vec{v}}{r - \frac{1}{c}\vec{v}\cdot\vec{r}}$$



$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r-\frac{1}{c}\vec{v}\cdot\vec{r}}$$

$$\vec{E}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}$$

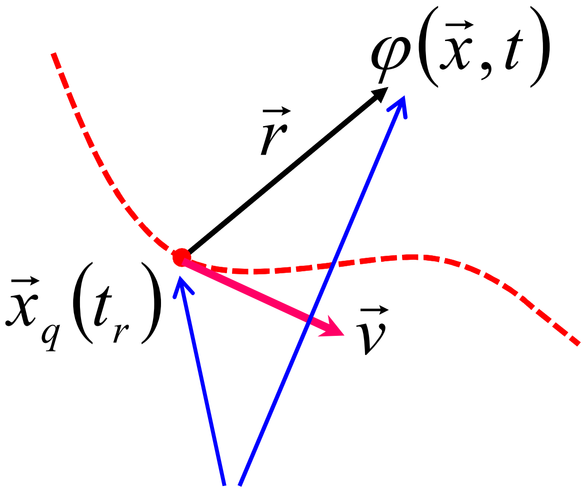
$$\vec{A}(\vec{x},t)=\frac{1}{4\pi\varepsilon_0 c^2}\cdot\frac{q\vec{v}}{r-\frac{1}{c}\vec{v}\cdot\vec{r}}$$

$$\vec{B}=\nabla\times\vec{A}$$

- 需要对势求有关 x 和 t 的导数，而等式右边是t_r的函数

$$\vec{r}=\vec{x}-\vec{x}_q(t_r),$$

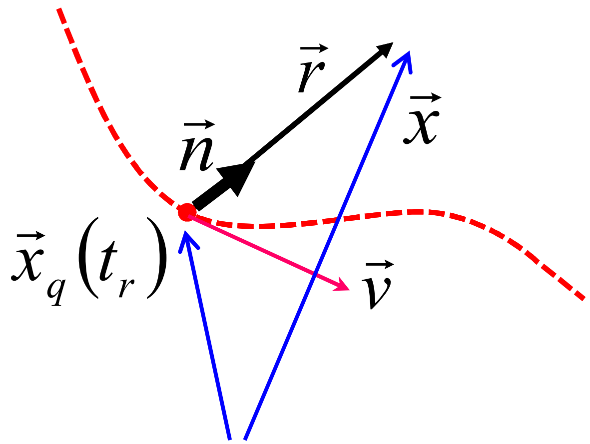
$$t_r=t-\frac{r}{c}$$



$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t},$$

$$\vec{B} = \nabla \times \vec{A}$$

$$t_r = t - \frac{r}{c}$$



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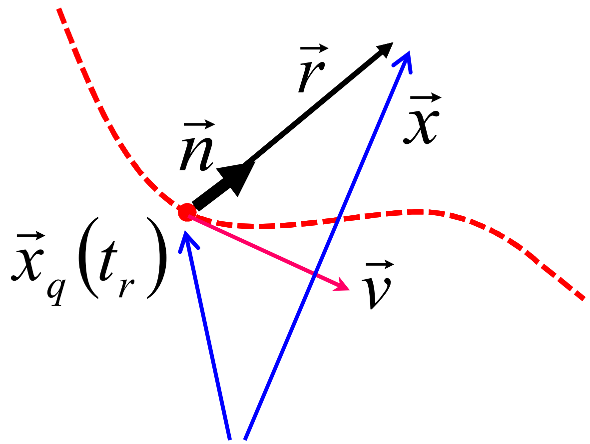


- 需要考察 $\partial t_r / \partial t = ?$

$$\nabla t_r = ?$$

$$\nabla r$$

$$\nabla(\vec{r} \cdot \vec{v})$$

- 先考察 $\partial t_r / \partial t = ?$

$$\vec{E} = -\nabla\varphi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$t_r = t - \frac{r}{c}$$

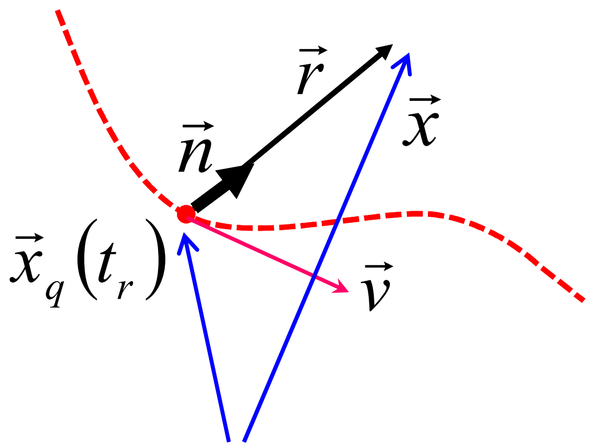
$$\Rightarrow \frac{\partial t_r}{\partial t} = 1 - \frac{1}{c}\cdot\frac{\partial r}{\partial t_r}\cdot\frac{\partial t_r}{\partial t}$$

$$\frac{\partial r}{\partial t_r} = \frac{\partial}{\partial t_r}\sqrt{[\vec{x}-\vec{x}_q(t_r)]^2}$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{[\vec{x}-\vec{x}_q(t_r)]^2}}\cdot\frac{\partial}{\partial t_r}[\vec{x}-\vec{x}_q(t_r)]^2$$

$$= \frac{1}{2}\cdot\frac{\vec{r}}{r}\cdot 2\frac{\partial\left[-\vec{x}_q(t_r)\right]}{\partial t_r} = -\frac{\vec{r}}{r}\cdot\vec{v},$$

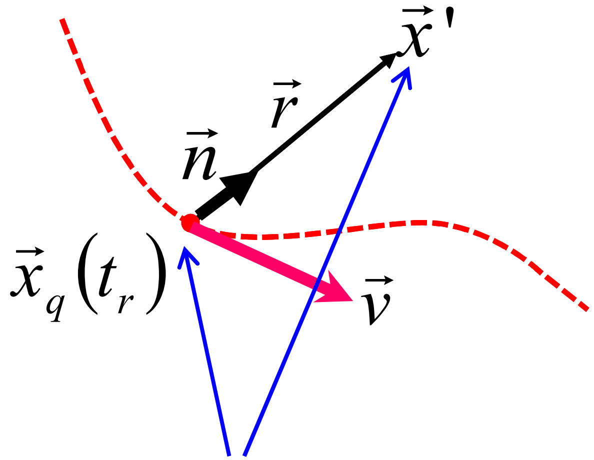
$$\frac{\partial r}{\partial t_r} = -\vec{v}\cdot\vec{n}$$



$$\frac{\partial r}{\partial t_r} = -\vec{v}\cdot\vec{n}$$

$$t_r = t - \frac{r}{c}$$

$$\frac{\partial t_r}{\partial t} = 1 - \frac{1}{c}\cdot\frac{\partial r}{\partial t_r}\cdot\frac{\partial t_r}{\partial t}$$



$$\frac{\partial t_r}{\partial t} = 1 - \frac{1}{c}\cdot\frac{\partial r}{\partial t_r}\cdot\frac{\partial t_r}{\partial t} = 1 + \frac{1}{c}\left(\vec{v}\cdot\vec{n}\right)\frac{\partial t_r}{\partial t}$$

$$\frac{\partial t_r}{\partial t}\cdot\left(1 - \frac{\vec{v}\cdot\vec{n}}{c}\right) = 1$$

得到：

$$\frac{\partial t_r}{\partial t} = \left(1 - \frac{\vec{v}\cdot\vec{n}}{c}\right)^{-1}$$

$$\nabla t_r = ?$$

$$\vec{r} = \vec{x} - \vec{x}_q(t_r)$$

$$t_r = t - \frac{r}{c}$$

$$\frac{\partial r}{\partial t_r} = -\vec{v}\cdot\vec{n}$$

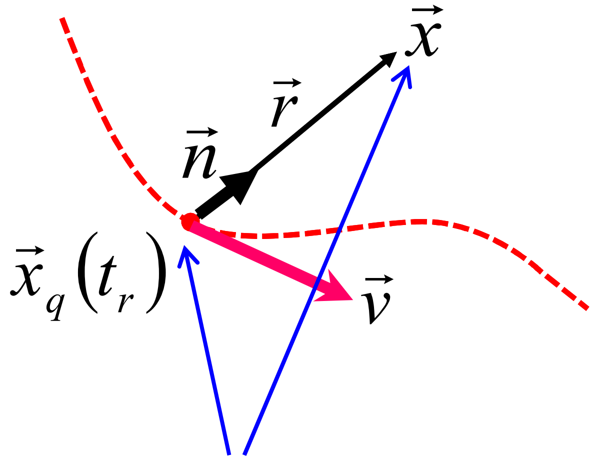
$$\nabla t_r = -\frac{1}{c}\nabla r = -\frac{1}{c}\nabla r\left(\vec{x}, t_r\right)$$

$$= -\frac{1}{c}\nabla r\Big|_{t_r=\text{常数}} - \frac{1}{c}\frac{\partial r}{\partial t_r}\nabla t_r$$

$$= -\frac{1}{c}\frac{\vec{r}}{r} + \frac{1}{c}\left(\vec{v}\cdot\vec{n}\right)\nabla t_r$$

$$(1 - \frac{\vec{v}\cdot\vec{n}}{c})\nabla t_r = -\frac{1}{c}\vec{n}$$

得到： $\nabla t_r = -(c - \vec{v}\cdot\vec{n})\vec{n}$



$$t_r = t - \frac{r}{c}$$

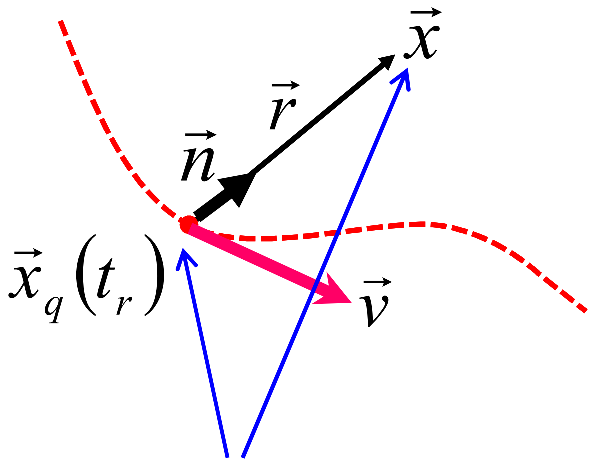
$$\nabla r \quad \Rightarrow \quad \nabla r = -c\nabla t_r$$

$$\nabla(\vec{r}\cdot\vec{v})$$

$$= (\vec{r}\cdot\nabla)\vec{v} + (\vec{v}\cdot\nabla)\vec{r} + \vec{r}\times(\nabla\times\vec{v}) + \vec{v}\times(\nabla\times\vec{r})$$ （教材附录I.23）

$$= \vec{v} + (\vec{r}\cdot\dot{\vec{v}} - v^2)\nabla t_r$$

（详细计算见课件的附录）



其中： $\dot{\vec{v}} = \dfrac{\mathrm{d}\vec{v}}{\mathrm{d}t_r}$ (accelration)

$$\phi(\vec{x},t)=\frac{1}{4\pi\varepsilon_0}\cdot\frac{q}{r-\frac{1}{c}\vec{v}\cdot\vec{r}}$$

(1) $\nabla\varphi$ 的计算

$$\varphi(\vec{x},t)=\frac{q}{4\pi\varepsilon_0}s^{-1},\quad\left(s\equiv r-\frac{\vec{r}\cdot\vec{v}}{c}\right)$$

$$\nabla\varphi=\frac{q}{4\pi\varepsilon_0}\nabla s^{-1}$$

$$t_r = t - \frac{r}{c}$$

$$\nabla \varphi = \frac{q}{4\pi\varepsilon_0} \nabla s^{-1}$$

$$\nabla r = -c \nabla t_r$$

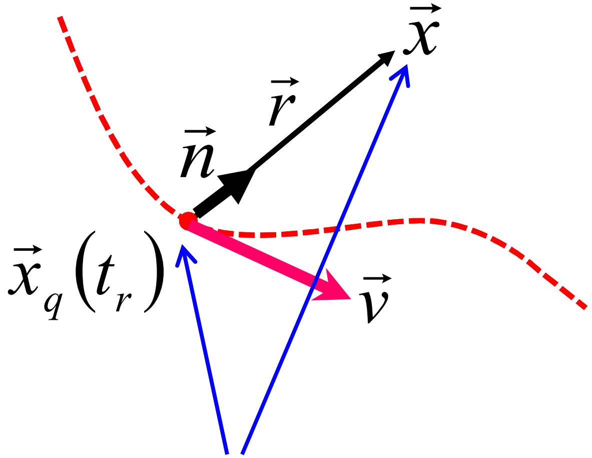
$$\nabla(\vec{r} \cdot \vec{v}) = \vec{v} + (\vec{r} \cdot \dot{\vec{v}} - v^2) \nabla t_r$$

$$\nabla s^{-1} = -s^{-2} \nabla \left(r - \frac{\vec{r} \cdot \vec{v}}{c}\right)$$

$$\left(s \equiv r - \frac{\vec{r} \cdot \vec{v}}{c} \right)$$

$$\nabla s^{-1} = -s^{-2} \left[-c \nabla t_r - \frac{\vec{v} + (\vec{r} \cdot \dot{\vec{v}} - v^2) \nabla t_r}{c} \right]$$

$$\nabla \varphi = -\frac{q}{4\pi\varepsilon_0} s^{-2} \left[-c \nabla t_r - \frac{\vec{v}}{c} - (\vec{r} \cdot \dot{\vec{v}} - v^2) \frac{\nabla t_r}{c} \right]$$



(2) $\dfrac{\partial \vec{A}}{\partial t}$ 的计算

$$\vec{A}(\vec{x},t)=\frac{q}{4\pi\varepsilon_0 c^2}\frac{\vec{v}}{s},$$

$$\left(s\equiv r-\frac{\vec{r}\cdot\vec{v}}{c}\right)$$

$$\frac{\partial(\vec{r}\cdot\vec{v})}{\partial t_r}=-\vec{v}\cdot\vec{v}+\vec{r}\cdot\dot{\vec{v}}$$

$$\vec{A}(\vec{x},t)=\frac{q}{4\pi\varepsilon_0 c^2}s^{-1}\vec{v},\quad \frac{\partial(\vec{r}\cdot\vec{v})}{\partial t_r}=-\vec{v}\cdot\vec{v}+\vec{r}\cdot\dot{\vec{v}}$$

$$t_r=t-\frac{r}{c}$$

$$\vec{r}=\vec{x}-\vec{x}_q(t_r)$$

$$\left(s\equiv r-\frac{\vec{r}\cdot\vec{v}}{c}\right)$$

$$\frac{\partial r}{\partial t_r}=-\vec{v}\cdot\vec{n}$$

$$\frac{\partial\vec{A}}{\partial t}=\frac{\partial\vec{A}}{\partial t_r}\cdot\frac{\partial t_r}{\partial t}$$

$$\frac{\partial\vec{A}}{\partial t_r}=\frac{q}{4\pi\varepsilon_0 c^2}\cdot\frac{\partial(s^{-1}\vec{v})}{\partial t_r}=\frac{q}{4\pi\varepsilon_0 c^2}\left[\frac{\partial s^{-1}}{\partial t_r}\vec{v}+s^{-1}\dot{\vec{v}}\right]$$

$$\frac{\partial s^{-1}}{\partial t_r}=-s^{-2}\frac{\partial s}{\partial t_r}=-s^{-2}\left[\frac{\partial r}{\partial t_r}-\frac{\partial(\vec{r}\cdot\vec{v})}{c\,\partial t_r}\right]$$

$$=-s^{-2}\left(-\vec{v}\cdot\vec{n}+\frac{\vec{v}\cdot\vec{v}-\vec{r}\cdot\dot{\vec{v}}}{c}\right)$$

$$\frac{\partial\vec{A}}{\partial t}=\frac{q}{4\pi\varepsilon_0 c^2}\left[-s^{-2}\vec{v}\left(-\vec{v}\cdot\vec{n}+\frac{\vec{v}\cdot\vec{v}-\vec{r}\cdot\dot{\vec{v}}}{c}\right)+s^{-1}\dot{\vec{v}}\right]\frac{\partial t}{\partial t_r}$$

(3) $\nabla \times \vec{A}$ 的计算

$$\vec{A} = \frac{q\vec{v}}{4\pi\varepsilon_0 c^2} s^{-1}, \qquad \left(s \equiv r - \frac{\vec{r} \cdot \vec{v}}{c} \right)$$

$$\nabla \times \vec{A} = \frac{q}{4\pi\varepsilon_0 c^2} \nabla \times \left(s^{-1} \vec{v} \right)$$

$$\nabla s^{-1} = -s^{-2}\left[-c\nabla t_r - \frac{\vec{v} + (\vec{r}\cdot\dot{\vec{v}} - v^2)\nabla t_r}{c}\right]$$

$$\nabla \times \vec{A} = \frac{\mu_0 q}{4\pi} \nabla \times \left(s^{-1}\vec{v}\right)$$

$$\Rightarrow \quad \nabla \times \left(s^{-1}\vec{v}\right) = (\nabla s^{-1}) \times \vec{v} + s^{-1}\nabla \times \vec{v}$$

$$\nabla \times \vec{v} = -\frac{\mathrm{d}\vec{v}}{\mathrm{d}t_r} \times \nabla t_r = -\dot{\vec{v}} \times \nabla t_r$$

$$\nabla \times \vec{A}$$

$$= \frac{q}{4\pi\varepsilon_0 c^2}\left\{-s^{-2}\left[-c\nabla t_r - \frac{\vec{v} + (\vec{r}\cdot\dot{\vec{v}} - v^2)\nabla t_r}{c}\right] \times \vec{v} - s^{-1}\dot{\vec{v}} \times \nabla t_r\right\}$$

$$\nabla\varphi = -\frac{q}{4\pi\varepsilon_0} s^{-2}\left[-c\nabla t_r - \frac{\vec{v}}{c} - (\vec{r}\cdot\dot{\vec{v}} - v^2)\frac{\nabla t_r}{c}\right]$$

$$\frac{\partial\vec{A}}{\partial t} = \frac{q}{4\pi\varepsilon_0 c^2}\left[-s^{-2}\vec{v}(-\vec{v}\cdot\vec{n} + \frac{\vec{v}\cdot\vec{v} - \vec{r}\cdot\dot{\vec{v}}}{c}) + s^{-1}\dot{\vec{v}}\right]\frac{\partial t}{\partial t_r}$$

$$\nabla\times\vec{A} = \frac{q}{4\pi\varepsilon_0 c^2}\left\{-s^{-2}\left[-c\nabla t_r - \frac{\vec{v}}{c} - (\vec{r}\cdot\dot{\vec{v}} - v^2)\frac{\nabla t_r}{c}\right]\times\vec{v} - s^{-1}\left(\dot{\vec{v}}\times\nabla t_r\right)\right\}$$

一）讨论当粒子速度v<<c 时所激发的电磁场

$$\nabla t_r = -\left(c - \vec{v}\cdot\vec{n}\right)^{-1}\vec{n}$$

$$\nabla t_r \approx -\frac{\vec{n}}{c}$$

$$\frac{\partial t_r}{\partial t} = \left(1 - \frac{\vec{v}\cdot\vec{n}}{c}\right)^{-1}$$

$$\frac{\partial t_r}{\partial t} \approx 1$$

$$\left(s \equiv r - \frac{\vec{r}\cdot\vec{v}}{c}\right)$$

$$s \equiv r - \frac{\vec{r}\cdot\vec{v}}{c} \approx r$$

$$s \approx r$$

$$\nabla t_r \approx -\frac{\vec{n}}{c}$$

$$\frac{\partial t_r}{\partial t} \approx 1$$

$$\nabla \varphi = -\frac{q}{4\pi\varepsilon_0} s^{-2} \left[-c\nabla t_r - \frac{\vec{v}}{c} - (\vec{r} \cdot \dot{\vec{v}} - v^2) \frac{\nabla t_r}{c} \right]$$

$$\approx -\frac{q}{4\pi\varepsilon_0} r^{-2} \left[\vec{n} - \frac{\vec{v}}{c} + (\vec{r} \cdot \dot{\vec{v}} - v^2) \frac{\vec{n}}{c^2} \right]$$

$$\nabla \varphi \approx -\frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \vec{n} - \frac{q}{4\pi\varepsilon_0} \frac{\vec{r} \cdot \dot{\vec{v}}}{r^2 c^2} \vec{n}$$

$$\nabla t_r \approx -\frac{\vec{n}}{c}$$

$$\frac{\partial t_r}{\partial t} \approx 1$$

$$s \approx r$$

$$\frac{\partial \vec{A}}{\partial t} = \frac{q}{4\pi\varepsilon_0 c^2}\left[-s^{-2}\left(-\vec{v}\cdot\vec{n} + \frac{\vec{v}\cdot\vec{v} - \vec{r}\cdot\dot{\vec{v}}}{c}\right)\vec{v} + s^{-1}\dot{\vec{v}}\right]\frac{\partial t}{\partial t_r}$$

$$\approx \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r^2}\left(-\vec{v}\cdot\vec{n} + \frac{\vec{v}\cdot\vec{v} - \vec{r}\cdot\dot{\vec{v}}}{c}\right)\vec{v} + \frac{1}{r}\dot{\vec{v}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r^2}\left(-\vec{v}\cdot\vec{n} + \frac{\vec{v}\cdot\vec{v} - \vec{r}\cdot\dot{\vec{v}}}{c}\right)\frac{\vec{v}}{c^2} + \frac{1}{r}\frac{\dot{\vec{v}}}{c^2}\right]$$

$$\frac{\partial \vec{A}}{\partial t} \approx \frac{q}{4\pi\varepsilon_0}\frac{\dot{\vec{v}}}{rc^2}$$

$$\nabla\varphi \approx -\frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} - \frac{q}{4\pi\varepsilon_0}\frac{\vec{n}}{r^2c^2}\vec{r}\cdot\dot{\vec{v}}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$$

$$\frac{\partial\vec{A}}{\partial t} \approx \frac{q}{4\pi\varepsilon_0}\frac{\dot{\vec{v}}}{rc^2}$$

$$\approx \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} + \frac{q}{4\pi\varepsilon_0}\frac{\vec{n}}{r^2c^2}\vec{r}\cdot\dot{\vec{v}} - \frac{q}{4\pi\varepsilon_0}\frac{\dot{\vec{v}}}{rc^2}$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} + \frac{q}{4\pi\varepsilon_0c^2}\left(\frac{\vec{n}}{r^2}\vec{r}\cdot\dot{\vec{v}} - \frac{\dot{\vec{v}}}{r}\right)$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} + \frac{q}{4\pi\varepsilon_0c^2}\frac{1}{r^3}\left[\vec{r}\left(\vec{r}\cdot\dot{\vec{v}}\right) - \dot{\vec{v}}\left(\vec{r}\cdot\vec{r}\right)\right]$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} + \frac{q}{4\pi\varepsilon_0c^2}\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{v}})$$

$$\vec{B} = ?$$

$$\nabla t_r \approx -\frac{\vec{n}}{c}$$

$$\frac{\partial t_r}{\partial t} \approx 1$$

$$s \approx r$$

$$\nabla \times \vec{A} = \frac{q}{4\pi\varepsilon_0 c^2}\left\{-s^{-2}\left[-c\nabla t_r - \frac{\vec{v}}{c} - (\vec{r}\cdot\dot{\vec{v}} - v^2)\frac{\nabla t_r}{c}\right]\times\vec{v} - s^{-1}\left(\dot{\vec{v}}\times\nabla t_r\right)\right\}$$

$$\approx \frac{q}{4\pi\varepsilon_0 c^2}\left\{\frac{-1}{r^2}\left[\vec{n} - \frac{\vec{v}}{c} + (\vec{r}\cdot\dot{\vec{v}} - v^2)\frac{\vec{n}}{c^2}\right]\times\vec{v} + \frac{1}{r}\dot{\vec{v}}\times\frac{\vec{n}}{c}\right\}$$

$$\approx \frac{q}{4\pi\varepsilon_0 c^2}\left(\frac{-1}{r^2}\vec{n}\times\vec{v} + \frac{1}{r}\dot{\vec{v}}\times\frac{\vec{n}}{c}\right)$$

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{q\vec{v}\times\vec{r}}{r^3} + \frac{q}{4\pi\varepsilon_0 c^3}\frac{\dot{\vec{v}}\times\vec{r}}{r^2}$$

在 v<<c 情况下：

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}\vec{n} + \frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{v}})$$

$$\vec{B} = \frac{1}{4\pi\varepsilon_0 c^2}\frac{q\vec{v}\times\vec{r}}{r^3} + \frac{q}{4\pi\varepsilon_0 c^3}\frac{\dot{\vec{v}}\times\vec{r}}{r^2}$$

结论： 在 v<<c 情况下：

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{v}})$$

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^3}\frac{\dot{\vec{v}}\times\vec{r}}{r^2}$$

静场项 Static term　　动力学项 Dynamic term

在 v<<c 情况下：

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{v}})$$

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^3}\frac{\dot{\vec{v}}\times\vec{r}}{r^2}$$

静场项$\propto r^{-2}$

- 对于匀速运动 ➔ 非辐射场

在 v<<c 情况下：

$$\vec{E}=\frac{q}{4\pi\varepsilon_0}\frac{\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^2}\frac{1}{r^3}\vec{r}\times(\vec{r}\times\dot{\vec{v}})$$

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{v}\times\vec{r}}{r^3}+\frac{q}{4\pi\varepsilon_0 c^3}\frac{\dot{\vec{v}}\times\vec{r}}{r^2}$$

辐射场 $\propto r^{-1}$

- 加速运动 ➔ 辐射场

附录： $\nabla(\vec{r}\cdot\vec{v})$ 的详细计算推导

$$\nabla(\vec{r}\cdot\vec{v})=(\vec{r}\cdot\nabla)\vec{v}+(\vec{v}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{v})+\vec{v}\times(\nabla\times\vec{r}) \tag{I.23}$$

第一项：

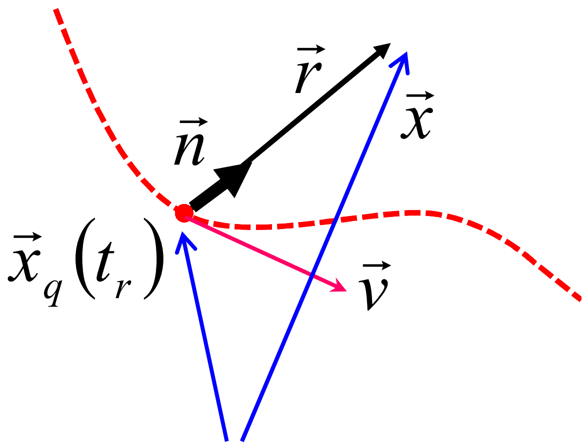
$$(\vec{r}\cdot\nabla)\vec{v}=\left(r_x\frac{\partial}{\partial x}+r_y\frac{\partial}{\partial y}+r_z\frac{\partial}{\partial z}\right)\vec{v}(t_r)$$

$$= \quad r_x\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial x}+r_y\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial y}+r_z\frac{d\vec{v}}{dt_r}\frac{\partial t_r}{\partial z}$$

$$=\dot{\vec{v}}(\vec{r}\cdot\nabla t_r)$$

$$\nabla(\vec{r}\cdot\vec{v})=(\vec{r}\cdot\nabla)\vec{v}+(\vec{v}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{v})+\vec{v}\times(\nabla\times\vec{r})$$

第二项： $(\vec{v}\cdot\nabla)\vec{r}=(\vec{v}\cdot\nabla)\vec{x}-(\vec{v}\cdot\nabla)\vec{x}_q(t_r)$



$$(\vec{v}\cdot\nabla)\vec{x}=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)(x,y,z)$$

$$=v_x\frac{\partial(x,y,z)}{\partial x}+v_y\frac{\partial(x,y,z)}{\partial y}+v_z\frac{\partial(x,y,z)}{\partial z}$$

$$=v_x(1,0,0)+v_x(0,1,0)+v_z(0,0,1)=(v_x,v_y,v_z)=\vec{v}$$

$$\vec{r}=\vec{x}-\vec{x}_q(t_r)$$

$$\frac{d\vec{x}_q}{dt_r}=\vec{v}$$

$$(\vec{v}\cdot\nabla)\vec{x}_q=\left(v_x\frac{\partial}{\partial x}+v_y\frac{\partial}{\partial y}+v_z\frac{\partial}{\partial z}\right)\vec{x}_q$$

$$=\ v_x\frac{d\vec{x}_q}{dt_r}\frac{\partial t_r}{\partial x}+v_y\frac{d\vec{x}_q}{dt_r}\frac{\partial t_r}{\partial y}+r_z\frac{d\vec{x}_q}{dt_r}\frac{\partial t_r}{\partial z}=\vec{v}(\vec{v}\cdot\nabla t_r)$$

$$\boxed{(\vec{v}\cdot\nabla)\vec{r}=\vec{v}-\vec{v}(\vec{v}\cdot\nabla t_r)}$$

$$\nabla(\vec{r}\cdot\vec{v})=\underline{(\vec{r}\cdot\nabla)}\vec{v}+\underline{(\vec{v}\cdot\nabla)}\vec{r}+\vec{r}\times(\underline{\nabla\times\vec{v}})+\vec{v}\times(\underline{\nabla\times\vec{r}})$$

第三项中：

$$\nabla\times\vec{v}=\nabla\times\vec{v}(t_r)=-\frac{d\vec{v}}{dt_r}\times\nabla t_r=-\dot{\vec{v}}\times\nabla t_r$$

第四项中：

$$\nabla\times\vec{r}=\nabla\times\vec{x}-\nabla\times\vec{x}_q=0-\nabla\times\vec{x}_q$$

$$=\frac{d\vec{x}_q}{dt_r}\times\nabla t_r=\vec{v}\times\nabla t_r$$

把上面的结果带入：

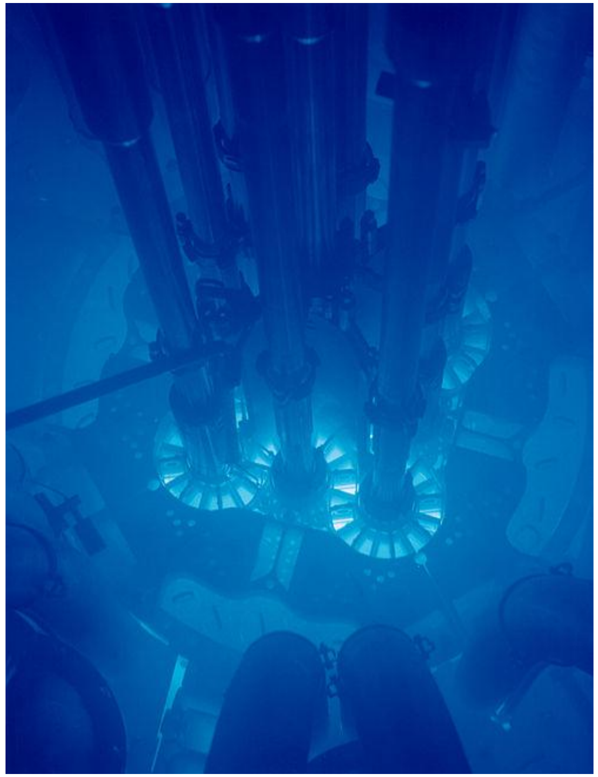
$$
\begin{aligned}
&\nabla(\vec{r}\cdot\vec{v})\\
&=(\vec{r}\cdot\nabla)\vec{v}+(\vec{v}\cdot\nabla)\vec{r}+\vec{r}\times(\nabla\times\vec{v})+\vec{v}\times(\nabla\times\vec{r})\\
&=\dot{\vec{v}}(\vec{r}\cdot\nabla t_r)+\vec{v}-\vec{v}(\vec{v}\cdot\nabla t_r)-\vec{r}\times(\dot{\vec{v}}\times\nabla t_r)+\vec{v}\times(\vec{v}\times\nabla t_r)\\
&=\dot{\vec{v}}(\vec{r}\cdot\nabla t_r)+\vec{v}-\vec{v}(\vec{v}\cdot\nabla t_r)-(\vec{r}\cdot\nabla t_r)\dot{\vec{v}}+(\vec{r}\cdot\dot{\vec{v}})\nabla t_r\\
&\qquad\qquad\qquad +\ (\vec{v}\cdot\nabla t_r)\vec{v}-(\vec{v}\cdot\vec{v})\nabla t_r
\end{aligned}
$$

约去相消项得：

$$
\nabla(\vec{r}\cdot\vec{v})=\vec{v}+(\vec{r}\cdot\dot{\vec{v}})\nabla t_r-(\vec{v}\cdot\vec{v})\nabla t_r
$$

所以：

$$
\nabla(\vec{r}\cdot\vec{v})=\vec{v}+(\vec{r}\cdot\dot{\vec{v}}-v^2)\nabla t_r
$$



Cherenkov radiation glowing in the core of the Advanced Test Reactor