Quantum Error Correction II

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Quantum error correction

Quantum code

A quantum code is a subspace of the N-qubit Hilbert space.

For a given subspace:

Choose an orthonormal, or basis $\{|\psi_i\rangle\}$

Use the projection onto the code space $\Pi = \sum_i |\psi_i\rangle\langle\psi_i|$

Now suppose the error of the system is characterized by the quantum noise $\mathcal{E} = \{E_k\}$, where E_ks are the Kraus operators.

Quantum error-correcting criteria

A quantum code with orthonormal basis $\{|\psi_i\rangle\}$ corrects the error set $\mathcal{E} = \{E_k\}$ if and only if

$$\langle\psi_i|E_k^\dagger E_l|\psi_j\rangle = c_{kl}\delta_{ij},$$

or in terms of Π

$$\Pi E_k^\dagger E_l \Pi = c_{kl}\Pi.$$

Quantum code distance

Consider an N-qubit operator O of the form

$$O = O_1 \otimes O_2, \ldots, \otimes O_N,$$

where each O_k acting on the kth qubit.
wt(O): the weight of M, i.e. the number of non-trivial O_ks.
Consider the depolarizing noise $\mathcal{E}_{DP}^{\otimes N}$, where we want a quantum code capable of correcting t-errors, it is enough to consider only Kraus operator M of weight $\leq t$ where each O_k are one of the Pauli operators $\{I, X_k, Y_k, Z_k\}$. In other words, a code is capable of correcting t errors for any O with weight $\leq 2t + 1$.

Quantum code distance

The distance for quantum code with orthonormal basis $\{|\psi_i\rangle\}$ is the largest possible weight d such that

$$\langle \psi_i | O | \psi_j \rangle = c_O \delta_{ij}$$

holds for all operators O with $\mathrm{wt}(O) < d$.

The stabilizer formalism

N-qubit Pauli operators:

$$O_1 \otimes O_2, \dots, \otimes O_N,$$

where each $O_k \in \{I_k, X_k, Y_k, Z_k\}$, is a Pauli operator acting on the kth qubit. All such N-qubit Pauli operators together form a group that we denote by $\mathcal{P}_N$.

The stabilizer formalism

Let $\mathcal{S} \subset \mathcal{P}_N$ be an abelian subgroup of the Pauli group that does not contain $-I$, and let

$$Q(\mathcal{S}) = \{|\psi\rangle \text{ s.t. } P|\psi\rangle = |\psi\rangle, \ \forall P \in \mathcal{S}\}.$$

Then $Q(\mathcal{S})$ is a stabilizer code and $\mathcal{S}$ is its stabilizer.

Let $\mathcal{S}^{\perp} = \{E \in \mathcal{P}_N, \text{ s.t. } [E, S] = 0, \ \forall S \in \mathcal{S}\}$.

Stabilizer code: dimension and distance

Let $\mathcal{S}$ be a stabilizer with $N - M$ generators. Then $\mathcal{S}$ encodes M qubits and has distance d, where d is the smallest weight of a Pauli operator in $\mathcal{S}^{\perp} \setminus \mathcal{S}$.

The five-qubit code

Consider the stabilizer

$$\mathcal{S} = \langle g_1, g_2, g_3, g_4 \rangle,$$

where

$$\begin{array}{rccccc} g_1 = & X & Z & Z & X & I \\ g_2 = & I & X & Z & Z & X \\ g_3 = & X & I & X & Z & Z \\ g_4 = & X & I & X & Z & Z \end{array}$$

The smallest weight operator in $\mathcal{S}^{\perp} \setminus \mathcal{S}$ has weight 3. So this code has length 5, dimension 2^1, and distance 3, denoted by $[[5, 1, 3]]$. The projection onto the code space

$$\Pi = \frac{1}{2^4} \prod_{i=1}^{4} (I + g_i)$$

The five-qubit code

$$
\begin{array}{rlllll}
g_1 = & X & Z & Z & X & I \\
g_2 = & I & X & Z & Z & X \\
g_3 = & X & I & X & Z & Z \\
g_4 = & X & I & X & Z & Z \\
\bar{Z} = & Z & Z & Z & Z & Z \\
\bar{X} = & X & X & X & X & X
\end{array}
$$

This defines $|0_L\rangle$ and $|1_L\rangle$.
$\Pi = \frac{1}{2^4} \prod_{i=1}^{4} (I + g_i)$, and

$$\Pi E_k^\dagger E_l \Pi = c_{kl} \Pi.$$

The code space spanned by $\{|0_L\rangle, |1_L\rangle\}$ is the ground state space of the Hamiltonian

$$H = -\sum_{i=1}^{4} g_i$$

The four-qubit code

Consider a length 4 code with stabilizer $\mathcal{S}$ generated by the following two Pauli operators.

$$\begin{array}{rcccc} g_1 = & X & X & X & X \\ g_2 = & Z & Z & Z & Z \end{array}$$

There are total $n = 4$ qubits and 2 generators for the stabilizer, so this code encodes $4 - 2 = 2$ qubits. The logical $|0_L\rangle|0_L\rangle$ can be chosen as the state stabilized by the following four Pauli operators.

$$\begin{array}{rcccc} g_1 = & X & X & X & X \\ g_2 = & Z & Z & Z & Z \\ \bar{Z}_1 = & I & Z & Z & I \\ \bar{Z}_2 = & I & I & Z & Z \end{array}$$

The distance of this code is 2, meaning that the smallest weight Pauli operator which commute with g_1, g_2 is 2, for instance, $\bar{Z}_1$ is such an operator with weight 2. Hence this is a $[[4, 2, 2]]$ code.

Stabilizer states

If a stabilizer code of N-qubit has N generators, then the dimension of the common eigenspace of eigenvalue 1 will be of dimension $2^{N-N} = 1$. That is, the stabilizer code contains indeed only a unique state. Such kind of state is called stabilizer state.

For example, the 4-qubit version of the GHZ state

$$|GHZ_4\rangle = \frac{1}{2}(|0000\rangle + |1111\rangle)$$

is a stabilizer state. To see why, consider the following 4 stabilizer generators

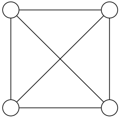
$$\begin{aligned} g_1 &= \quad Z \quad Z \quad I \quad I \\ g_2 &= \quad I \quad Z \quad Z \quad I \\ g_3 &= \quad I \quad I \quad Z \quad Z \\ g_4 &= \quad X \quad X \quad X \quad X, \end{aligned}$$

and it is straightforward to check that $g_i|GHZ\rangle_4 = |GHZ\rangle_4$.

Graph states

There is a special kind of stabilizer states called the graph states, whose stabilizer generators correspond to some given graphs. We start from an undirected graph G with n-vertices. For the ith vertex, we associate it with a stabilizer generator

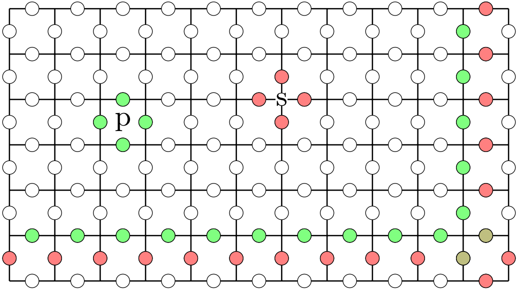
$$g_i = X_i \bigotimes_{k \in \text{neighbor } i} Z_k,$$

For a 4-qubit complete graph , the 4 stabilizer generators are given by

$$\begin{array}{rcccc} g_1 = & X & Z & Z & Z \\ g_2 = & Z & X & Z & Z \\ g_3 = & Z & Z & X & Z \\ g_4 = & Z & Z & Z & X \end{array}$$

Toric code

The square lattice

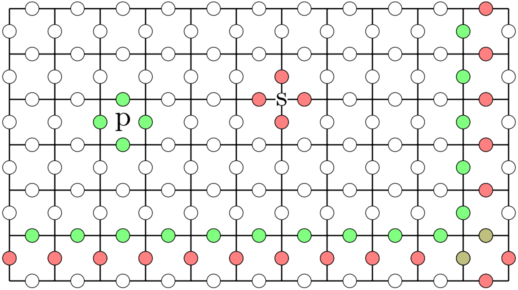


There are two types of stabilizer generators.

Type I (Star type): $A_s^Z = \prod_{j \in star(s)} Z_j$
Type II (Plaquette type): $A_p^X = \prod_{j \in plaquette(p)} X_j$

$$\prod_s A_s^Z = \prod_p A_p^X = I$$

Code distance



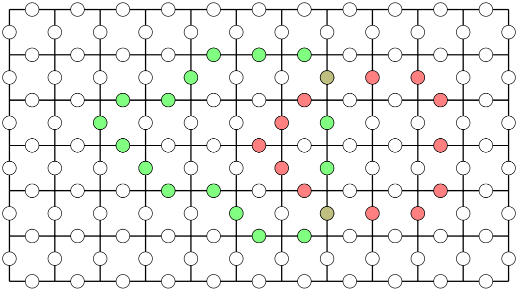
Total $2r^2$ qubits, but $r^2 + r^2 - 2 = 2r^2 - 2$ stabilizer generators. So the code has dimension 2^2.

The logical operators are cycles on the torus, hence the distance of the code is r.

The Hamiltonian

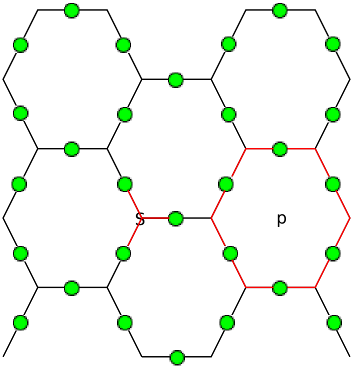
$$
\begin{aligned}
H_{toric} &= -\sum_s A_s^Z - \sum_p A_p^X \\
&= -\sum_s \prod_{j \in star(s)} Z_j - \sum_p \prod_{p \in plaquette(p)} X_j .
\end{aligned}
$$

A ground state $|\psi_g\rangle = \sum_{g \in \mathcal{S}_X} g|0\rangle^{\otimes 2r^2}$.



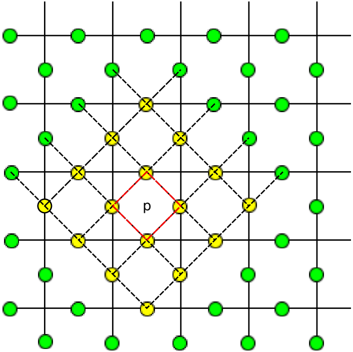
Properties

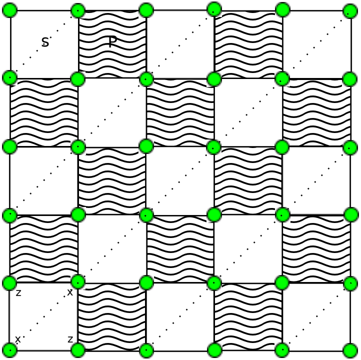
· Every stabilizer generator is local
· The code space encodes two qubits (i.e. four-dimensional subspace)
· The code distance grows with r, as an order of $\sqrt{n}$ when n goes arbitrarily large



The Wen-plaquette model

$$H_{wp} = -\sum_{ij} X_{i,j} Z_{i,j+1} X_{i+1,j+1} Z_{i+1,j}.$$





For any qubit i on any of the diagonal dashed lines, perform

$$X_i \leftrightarrow Z_i$$

Codeword stabilized (CWS) quantum code

Recall the classical repetition code

$$0 \to 000, \quad 1 \to 111$$

This code has $d = 3$: corrects one error, or detects two errors.

$$\mathcal{E} = \{100, 010, 001, 101, 011, 110\}$$

Error detection condition

The code $\mathcal{C}$ detects error set $\mathcal{E}$ iff

$$\mathbf{c}_i \neq \mathbf{c}_j \oplus \mathbf{e}, \quad \forall \mathbf{c}_i, \mathbf{c}_j \in \mathcal{C}, \quad \forall \mathbf{e} \in \mathcal{E}.$$

Codeword stabilized (CWS) quantum code

Ingredient 1: a graph G of n vertices

Ingredient 2: a binary classical code $\mathcal{C}$

$\mathcal{C}$ detects errors induced by G

Codeword stabilized (CWS) quantum code

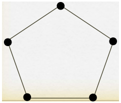
$((n, K, d))$: length n, dimension K, distance d

Ingredient 1: a graph G of n vertices $G \leftrightarrow |G\rangle$

Ingredient 2: a binary classical code $\mathcal{C} \in \{0,1\}^n$

Basis for quantum code

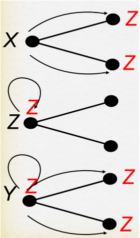
$$|\psi_i\rangle = Z^{\mathbf{c}_i}|G\rangle, \quad \mathbf{c}_i \in \mathcal{C}$$



$$\mathcal{C} = \{00000, 11111\}.$$

$$|\psi_0\rangle = IIIII|G\rangle, \quad |\psi_1\rangle = ZZZZZ|G\rangle.$$

The X-Z rule



On a graph state X errors are equivalent to (possibly multiple) Z errors: the X-Z rule. We call these the 'induced' errors.

$$X_i|\psi_j\rangle = X_i Z^{\mathbf{c}_j}|G\rangle = X_i Z^{\mathbf{c}_j} g_i|G\rangle$$

where $g_i = X_i Z^{\text{neighbor}(i)}$. Therefore

$$X_i|\psi_j\rangle = \pm Z^{\text{neighbor}(i)}|\psi_j\rangle,$$

and note that $\langle\psi_i|E|\psi_j\rangle = 0$.

X-Z rule: $X_i \rightarrow Z^{\text{neighbor}(i)}$, $Y_i \rightarrow Z^{\text{neighbor}(i)} Z_i$.

Error detection conditions

Since all induced errors are Zs, things are essentially classical. To detect errors from a set $\mathcal{E}$,

$$\langle\psi_i|E|\psi_j\rangle = 0, \ \forall E \in \mathcal{E}.$$

For basis of the form

$$|\psi_i\rangle = Z^{\mathbf{c}_i}|G\rangle, \quad \mathbf{c}_i \in \mathcal{C},$$

the condition becomes

$$\langle G|Z^{\mathbf{c}_i} E Z^{\mathbf{c}_j}|G\rangle = 0.$$

X-Z rule: $\forall E \in \mathcal{E} \rightarrow Z^{\mathbf{e}}$.

$$E|\psi_i\rangle = Z^{\mathbf{e}}|\psi_i\rangle = Z^{\mathbf{e}} Z^{\mathbf{c}_i}|G\rangle = Z^{\mathbf{c}_i \oplus \mathbf{e}}|G\rangle$$

Error detection conditions

$$\langle G|Z^{\mathbf{c}_i} E Z^{\mathbf{c}_j}|G\rangle = 0.$$
$$\Rightarrow \langle G|Z^{\mathbf{c}_i \oplus \mathbf{e} \oplus \mathbf{c}_j}|G\rangle = 0.$$

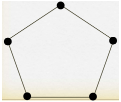
Based on the property of graph states, this holds if and only if

$$\mathbf{c}_i \neq \mathbf{e} \oplus \mathbf{c}_j.$$

This is nothing but the classical error detection condition.

$\mathcal{Q} = \{Z^{\mathbf{c}_i}|G\rangle\}$ detects errors set $\mathcal{E}$ $\Leftrightarrow$
$\mathcal{C} = \{\mathbf{c}_i\}$ detects the induced error set given by the graph G

Example: the $((5,2,3))$ code



$d=3$ need to detect double errors
$\mathcal{C}=\{00000,11111\}$.

$$|\psi_0\rangle = IIIII|G\rangle, \quad |\psi_1\rangle = ZZZZZ|G\rangle.$$

The error set

$$Z:\{10000,01000,00100,00010,00001\}$$

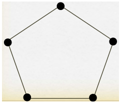
$$X:\{01001,10100,01010,00101,10010\}$$

$$Y:\{11001,11100,01110,00111,10011\}$$

Need to show

$$00000 \neq \text{two errors} \ \oplus \ 11111$$

Example: the $((5,6,2))$ code



$d = 3$ need to detect single errors

The error set

$$Z : \{10000, 01000, 00100, 00010, 00001\}$$

$$X : \{01001, 10100, 01010, 00101, 10010\}$$

$$Y : \{11001, 11100, 01110, 00111, 10011\}$$

$$\mathcal{C} = \{ \quad 00000, 11010, 01101 \\ 10110, 01011, 10101\}$$

$$\mathbf{c}_i \neq \mathbf{e} \oplus \mathbf{c}_j$$